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Unit roots, nonlinearities and structural breaks[1]

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Abstract

One of the most influential research fields in econometrics over the past decades concerns unit root testing in economic time series. In macro-economics much of the interest in the area originate from the fact that when unit roots are present, then shocks to the time series processes have a persistent effect with resulting policy implications. From a statistical perspective on the other hand, the presence of unit roots has dramatic implications for econometric model building, estimation, and inference in order to avoid the so-called spurious regression problem. The present paper provides a selective review of contributions to the field of unit root testing over the past three decades. We discuss the nature of stochastic and deterministic trend processes, including break processes, that are likely to affect unit root inference. A range of the most popular unit root tests are presented and their modifications to situations with breaks are discussed. We also review some results on unit root testing within the framework of non-linear processes.

1. Introduction

It is widely accepted that many time series in economics and finance exhibit trending behavior in the level (or mean) of the series. Typical examples include asset prices, exchange rates, real GDP, real wage series and so forth. In a recent paper White and Granger (2011) reflect on the nature of trends and make a variety of observations that seem to characterize these. Interestingly, as also noted by Phillips (2005), even though no one understands trends everybody still sees them in the data. In economics and other disciplines, almost all observed trends involve stochastic behaviour and purely deterministic trends are rare. However, a combination of stochastic and deterministic elements including structural changes seems to be a model class which is likely to describe the data well. Potentially the series may contain nonlinear features and even the apparent deterministic parts like level and trend may be driven by an underlying stochastic process that determines the timing and the size of breaks.

In recent years there has been a focus on stochastic trend models caused by the presence of unit roots. A stochastic trend is driven by a cumulation of historical shocks to the process and hence each shock will have a persistent effect. This feature does not necessarily characterize other types of trends where the source of the trend can be different and some or all shocks may only have a temporary effect. Time series with structural changes and unit roots share similar features that makes it difficult to discriminate between the two fundamentally different classes of processes. In principle, a unit root (or difference stationary) process can be considered as a process where each point in time has a level shift. On the other hand, if a time series process is stationary but is characterized by infrequent level shifts, certain

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(typically large) shocks tend to be persistent whereas other shocks have only a temporary influence. Many stationary nonlinear processes contain features similar to level shifts and unit root processes. Some types of regime switching models belong to this class of processes.

It is not surprising that discriminating between different types of trend processes is difficult. Still, there is an overwhelming literature which has focussed on unit root processes and how to distinguish these from other trending processes, and there are several reasons for this. One reason is the special feature of unit root processes regarding the persistence of shocks which may have important implications for the formulation of economic models and the measurement of impulse responses associated with economic policy shocks. Another reason concerns the fact that the presence of unit roots can result in spurious inference and hence should be appropriately accounted for in order to make valid inference when analyzing multivariate time series. The development of the notion of cointegration by Granger (1981, 1983) and Engle and Granger (1987) shows how time series with stochastic trends can be represented and modelled to avoid spurious relations. This field has grown tremendously since the initial contributions. For the statistical theory and overview, see Johansen (1995).

The purpose of the present chapter is to review recent advances and the current status in the field of unit root testing when accounting for deterministic trends, structural breaks and nonlinearities. We shall also consider some of the difficulties that arise due to other special features that complicate inference. There is a vast literature on these topics. Review articles include Haldrup and Jansson (2006) who focus on the size and power of unit root tests, and Perron (2006) who deals with structural breaks in stationary and non-stationary time series models. Other general overviews of unit root testing can be found in Stock (1994), Maddala and Kim (1998), and Phillips and Xiao (1998). The present review updates the present state of the art and includes a number of recent contributions in the field.

In section 2 we introduce a general class of basic processes where the focus is on unit root processes that can be mixed with the presence of deterministic components that potentially may exhibit breaks. In section 3 we review existing unit root tests that are commonly used in practice, i.e. the augmented Dickey-Fuller, Phillips and Perron, and the trinity of M-class of tests suggested by Perron and Ng (1996). We also briefly touch upon the literature on the design of optimal tests for the unit root hypothesis. The following two sections extends the analysis to the situation where the time series have a linear trend or drift and the initial condition is likely to affect inference. In particular, we address testing when there is general uncertainty about the presence of trends and the size of the initial condition. Section 6 extends the analysis to unit root testing in the presence of structural break processes for the cases where the break date is either known or unknown. Section 7 is concerned with unit root testing in nonlinear models followed by a section on unit root testing when the data exhibit particular features such as being bounded by their definition or exhibiting trends in both the levels and growth rates of the series. The paper finishes with an empirical illustration.

There are numerous relevant research topics which for space reasons we cannot discuss in this presentation. These include the literature on the design of optimal tests for the unit root hypothesis but also the highly relevant area of using the bootstrap in non-standard situations where existing procedures are likely to fail due to particular features of the data.

2. Trends in time series

We begin by reviewing some of the basic properties of unit root and trend-stationary processes including the possible structural breaks in such processes. Consider $T+1$ observations from the time series process generated by

$$y_t = f(t) + u_t, \qquad t = 0, 1, 2, ..., T \tag{1}$$

where

$$(1-\alpha L)u_t = C(L)\varepsilon_t,$$

u_t is a linear process with $\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma_\varepsilon^2)$ and $C(L) = \sum_{j=0}^{\infty} c_j L^j$, $\sum_{j=1}^{\infty} j|c_j| < \infty$, and $c_0 = 1$. L is the lag operator, $Lx_t = x_{t-1}$. $f(t)$ is a deterministic component to be defined later. When $\alpha = 1$ the series contains a unit root and a useful decomposition due to Beveridge and Nelson (1981) reads

$$\Delta u_t = C(1)\varepsilon_t + \Delta C^*(L)\varepsilon_t,$$

where $C(1) \neq 0$ and $C^*(L)$ satisfies requirements similar to those of $C(L)$. With this representation

$$y_t = y_0 + f(t) + C(1)\sum_{j=1}^{t} \varepsilon_j + \sum_{j=1}^{t-1} c_j^* \varepsilon_{t-j} \tag{2}$$

where $\tau_t = C(1)\sum_{j=1}^{t} \varepsilon_j$ is a stochastic trend component and $C^*(L)\varepsilon_t$ is a stationary component.

When u_t has no unit root, $|\alpha| < 1$, and setting $v_t = C(L)\varepsilon_t$, the process reads

$$\begin{aligned} y_t &= f(t) + (1-\alpha L)^{-1} v_t \\ &= \alpha^t y_0 + f(t) + \sum_{j=0}^{t-1} \alpha^j v_{t-j}. \end{aligned} \tag{3}$$

Equations (2) and (3) encompass many different features of unit root and trend stationary processes. As seen from (2), the presence of a unit root means that shocks will have a permanent effect and the level of the series is determined by a stochastic trend component in addition to the trend component $f(t)$. In principle, each period is characterized by a level shift through the term $\sum_{j=0}^{t-1} v_{t-j}$. In the trend stationary case $|\alpha| < 1$ shocks will only have a temporary effect, but each period also has a level shift through the deterministic component $f(t)$.

The models point to many of the statistical difficulties concerned with unit root testing in practice and the complications to discriminate between the different types of processes. For instance, in (2) and (3) we have not made any assumptions regarding the initial condition. This could be assumed fixed, or it could be stochastic in a certain way. However, the assumptions made are not innocuous with respect to the properties of unit root tests as we shall discuss later. The presence of deterministic components may also cause problems since deterministic terms can take many different forms. For instance the trend function can be linear in the parameters: $f(t) = d_t'\mu$ where d_t is a $k-$vector, for instance an intercept, a linear trend, and possibly a quadratic where $d_t' = (1, t, t^2)$, and μ is an associated parameter vector. Moreover, these different terms could have parameters that change over time within the sample. For example, the trend function may include changes in the level, the slope, or both, and these structural breaks may be at known dates (in which case the trend is still linear in parameters) or the break time may be generated according to a stochastic process, e.g. a Markov switching process.

Other difficulties concern the assumptions about the nature of the innovations governing the process. Generally, the short run dynamics of the process are unknown, the innovation variance may be heteroscedastic, and ε_t need not be Gaussian. We are going to address many of these complications and how to deal with these in practice. First, we want to consider a range of specifications of the trend function $f(t)$ that are essential for practical unit root testing.

2.1 Assumptions about the deterministic component $f(t)$

Linear trend. Following the empirical analysis of Nelson and Plosser (1982) it has been commonplace to consider the unit root model against one containing a linear trend. Fundamentally, the question asked is whether the trending feature of the data can be best described as a trend that never changes versus a trend that changes in every period. Assume that $f(t) = \mu + \beta t$ is a linear-in-parameters trend, in which case (3) becomes

$$y_t = \mu + \beta t + (1 - \alpha L)^{-1} C(L) \varepsilon_t \tag{4}$$

and

$$\Delta y_t = (\alpha - 1) y_{t-1} + \mu(1 - \alpha) + \alpha\beta + (1 - \alpha)\beta t + C(L)\varepsilon_t. \tag{5}$$

By comparing (4) and (5) it is seen that the role of deterministic components is different in the levels and the first differences representations. When a unit root is present, $\alpha = 1$, the constant term $\mu(1 - \alpha) + \alpha\beta = \beta$ in (5) represents the drift, whereas the slope $(1 - \alpha)\beta = 0$. This shows the importance of carefully interpreting the meaning of the deterministic terms under the null and the alternative hypothesis. Note that when $\beta \neq 0$, the linear trend will dominate the series even in the presence of a stochastic trend component.

Structural breaks. As emphasized by Perron (2006), discriminating between trends that either change every period or never change can be a rather rigid distinction. In many situations a more appropriate question could be whether a trend changes at every period or whether it only changes occassionally. This line of thinking initiated research by Rappoport and Reichlin (1989) and Perron (1989, 1990) who considered the possibility of certain events having a particularly strong impact on trends. Examples could include the Great Depression, World War II, the oil crises in the 1970s and early 1980s, the German reunification in 1990, the recent financial crisis initiated in 2007 and so forth. In modelling, such events may be ascribed stochastic shocks but possibly of a different nature than shocks occurring each period. The former are thus likely to be drawn from a different statistical distribution than the latter.

Perron (1989, 1990) suggested a general treatment of the structural break hypothesis where four different situations were considered that allowed a single break in the sample: (a) a change in the level, (b) a change in the level in the presence of a linear trend, (c) a change in the slope and (d) a change in both the level and slope. In implementing these models, two different transition mechanisms were considered following the terminology of Box and Tiao (1975); one is labeled the *additive outlier (AO) model* where the transition is instantaneous and the trend break function is linear in parameters, and one is labeled the *innovation outlier (IO) model* where changes occur via the innovation process and hence a gradual adjustment of a "big" shock takes place in accordance with the general dynamics of the underlying series. We will consider hypothesis testing later on and just note that distinguishing between these classes of break processes is important in the design of appropriate testing procedures.

Additive outlier models with level shift and trend break. Define the dummy variables DU_t and DT_t such that $DU_t = 1$ and $DT_t = t - T_1$ for $t > T_1$ and zero otherwise. The dummy variables allow various breaks to occur at time T_1. Using the classifications of Perron (2006) the following four AO-models are considered:

$$\begin{array}{ll} AO^a: & y_t = \mu_1 + (\mu_2 - \mu_1)DU_t + (1-\alpha L)^{-1}C(L)\varepsilon_t \\ AO^b: & y_t = \mu_1 + \beta t + (\mu_2 - \mu_1)DU_t + (1-\alpha L)^{-1}C(L)\varepsilon_t \\ AO^c: & y_t = \mu_1 + \beta_1 t + (\beta_2 - \beta_1)DT_t + (1-\alpha L)^{-1}C(L)\varepsilon_t \\ AO^d: & y_t = \mu_1 + \beta_1 t + (\mu_2 - \mu_1)DU_t + (\beta_2 - \beta_1)DT_t + (1-\alpha L)^{-1}C(L)\varepsilon_t. \end{array}$$

We assume that $\beta_1 \neq \beta_2$ and $\mu_1 \neq \mu_2$. Notice that under the null of a unit root, the differenced series reads $\Delta y_t = \Delta f(t) + C(L)\varepsilon_t$ where $\Delta f(t)$ takes the form of an impulse at time T_1 for the AO^a and AO^b models, whereas the AO^c model will have a level shift and the AO^d model will have a level shift plus an impulse blip at the break date.

Innovation change level shift and trend break models. The nature of these models depends on whether a unit root is present or absent. Assume initially that $|\alpha| < 1$. Then the IO-models read

$$\begin{array}{ll} IO^a: & y_t = \mu + (1-\alpha L)^{-1}C(L)(\varepsilon_t + \theta DU_t) \\ IO^b: & y_t = \mu + \beta t + (1-\alpha L)^{-1}C(L)(\varepsilon_t + \theta DU_t) \\ IO^d: & y_t = \mu + \beta t + (1-\alpha L)^{-1}C(L)(\varepsilon_t + \theta DU_t + \gamma DT_t). \end{array}$$

Hence, the impulse impact of a change in the intercept at time T_1 is given by θ and the long-run impact by $\theta(1-\alpha)^{-1}C(1)$. Similarly, the immediate impact of a change in slope is given by γ with long run impact $\gamma(1-\alpha)^{-1}C(1)$. Note that these models have similar characteristics to those of the AO models apart from the temporal dynamic adjustments of the IO models. Note that model c is not considered in the IO case because linear estimation methods cannot be used and will cause difficulties for practical applications.

Under the null hypothesis of a unit root, $\alpha = 1$, the meaning of the breaks in the IO models will cumulate unintentionally to higher order deterministic processes. It will therefore be necessary to redefine the dummies in this case whereby the models read

$$\begin{array}{ll} IO^{a_0}: & y_t = y_{t-1} + C(L)(\varepsilon_t + \delta(1-L)DU_t) \\ IO^{b_0}: & y_t = y_{t-1} + \beta + C(L)(\varepsilon_t + \delta(1-L)DU_t) \\ IO^{d_0}: & y_t = y_{t-1} + \beta + C(L)(\varepsilon_t + \delta(1-L)DU_t + \eta DU_t). \end{array}$$

where $(1-L)DU_t$ is an impulse dummy and $(1-L)DT_t = DU_t$. The impulse impact on the level of the series is given by δ and the long-run impact is by $\delta C(1)$, whereas for the IO^{d_0} model the impulse impact on the trend slope is given by η with long run slope equal to $\eta C(1)$. Qualitatively, the implications of the various break processes are thus similar to each other under the null and the alternative hypothesis.

2.2 Some other examples of trends

It is obvious from the examples above that in practice it can be difficult to discriminate between unit root (or difference stationary) processes and processes that are trend stationary with a possibly changing trend or level shifts. A stochastic trend has (many) innovations that tend to persist. A break process implies the existence of "big" structural breaks that tend to have a persistent effect. It goes without saying that distinguishing between these fundamentally different processes is even harder when one extends the above illustrations to cases with multiple breaks which potentially are generated by a stochastic process like a Bernoulli or Markov regime switching process.

As an example of this latter class of models, consider the so-called "mean-plus-noise" model in state space form, see e.g. Diebold and Inoue (2001):

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t \\ \mu_t &= \mu_{t-1} + v_t \\ v_t &= \begin{cases} 0 & \text{with probability } (1-p) \\ w_t & \text{with probability } p \end{cases} \end{aligned}$$

where $w_t \overset{i.i.d.}{\sim} N(0, \sigma_w^2)$ and $\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma_\varepsilon^2)$. Such a process consists of a mixture of shocks that have either permanent or transitory effect. If p is relatively small, then y_t will exhibit infrequent level shifts. Asymptotically, such a process which is really a generalization of the additive outlier level shift model AO^a, will behave like an $I(1)$ process. Granger and Hyung (2004) consider a related process in which the switches caused by a latent Markov chain have been replaced by deterministic breaks. A similar feature characterizes the STOPBREAK model of Engle and Smith (1999). A special case of the Markov-switching process of Hamilton (1989) behaves asymptotically as an $I(0)$ process: it has the same stationarity condition as a linear autoregressive model, but due to a Markov-switching intercept, it can generate a very high persistence in finite samples and can be difficult to discriminate from unit root processes, see e.g. Timmermann (2000) and Diebold and Inoue (2001). The intercept-switching threshold autoregressive process of Lanne and Saikkonen (2005) has the same property. It is a weakly stationary process but generates persistent realisations. Yet another model of the same type is the nonlinear sign model by Granger and Teräsvirta (1999) that is stationary but has 'misleading linear properties'. This means that autocorrelations estimated from realisations of this process show high persistence, which may lead the practitioner to think that the data have been generated by a nonstationary (long-memory), i.e., linear, model.

It is also possible to assume a deterministic intercept and generate realisations that have 'unit root properties'. The Switching-Mean Autoregressive model by González and Teräsvirta (2008) may serve as an example. In that model the intercept is characterised by a linear combination of logistic functions of time, which make both the intercept and with it the model quite flexible.

3. Unit root testing without deterministic components

In this section we will present unit root tests that are parametric or semiparametric extensions of the Dickey-Fuller test, see Dickey and Fuller (1979). We will state the underlying assumptions and consider generalizations in various directions.

3.1 The Dickey-Fuller test

Historically, the Dickey-Fuller test initiated the vast literature on unit root testing. Let us consider the case when (1) takes the simplified form of a AR(1) process

$$y_t = \alpha y_{t-1} + \varepsilon_t \tag{6}$$

where we assume that the initial observation is fixed at zero and $\varepsilon_t \sim i.i.d(0, \sigma_\varepsilon^2)$. The hypothesis to be tested is $H_0 : \alpha = 1$, and is tested against the one-side alternative $H_1 : \alpha < 1$. The least squares estimator of α reads

$$\widehat{\alpha} = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2}.$$

The associated $t-$statistic of the null hypothesis is

$$t_\alpha = \frac{\widehat{\alpha} - 1}{s/\sqrt{\sum_{t=1}^{T} y_{t-1}^2}}$$

where $s^2 = \frac{1}{T}\sum_{t=1}^{T} \left(y_t - \widehat{\alpha} y_{t-1}\right)^2$ is the estimate of the residual variance. Under the null hypothesis it is well known that these quantities have non-standard asymptotic distributions. In particular,

$$T(\widehat{\alpha} - 1) \xrightarrow{d} \frac{\int_0^1 W(r)dW(r)}{\int_0^1 W^2(r)dr} \tag{7}$$

and

$$t_\alpha \xrightarrow{d} \frac{\int_0^1 W(r)dW(r)}{\sqrt{\int_0^1 W^2(r)dr}} \tag{8}$$

where $W(r)$ is a Wiener process (or standard Brownian motion) defined on the unit interval and " $\xrightarrow{d}$ " indicates convergence in distribution. These distributions are often referred to as the Dickey-Fuller distributions even though they can be traced back to White (1958).

3.2 The Augmented Dickey-Fuller test

Because rather strict assumptions have been made regarding model (6) the limiting distributions (7) and (8) do not depend upon nuisance parameters under the null, i.e. the distributions are asymptotically pivotal. In particular, the assumption that innovations are *i.i.d.* is restrictive and a violation will mean that the relevant distributions are not as indicated above. To see this, assume that

$$y_t = \alpha y_{t-1} + u_t \tag{9}$$

where $u_t = C(L)\varepsilon_t$ with $C(L)$ satisfying the properties given in (1). We also assume that $y_0 = 0$.

This model allows more general assumptions regarding the serial correlation pattern of $y_t - \alpha y_{t-1}$ compared to the AR(1) model (6). Phillips (1987) shows that under these assumptions, the distributions (7) and (8) are modified as follows:

$$T(\widehat{\alpha} - 1) \xrightarrow{d} \frac{\int_0^1 W(r)dW(r) + \lambda}{\int_0^1 W^2(r)dr} \tag{10}$$

and

$$t_\alpha \xrightarrow{d} \frac{\omega}{\sigma} \frac{\int_0^1 W(r)dW(r) + \lambda}{\sqrt{\int_0^1 W^2(r)dr}} \tag{11}$$

where $\lambda = \left(\omega^2 - \sigma^2\right) / \left(2\omega^2\right)$, $\sigma^2 = E\left[u_t^2\right] = \sigma_\varepsilon^2 \left(\sum_{j=0}^{\infty} c_j^2\right)$ is the variance of u_t, and $\omega^2 = \lim_{T\to\infty} T^{-1} E\left[\left(\sum_{t=1}^{T} u_t\right)^2\right] = \sigma_\varepsilon^2 \left(\sum_{j=0}^{\infty} c_j\right)^2$ is the "long-run variance" of u_t. In fact, $\omega^2 = 2\pi f_u(0)$ where $f_u(0)$ is the spectral density of u_t evaluated at the origin. When the innovations are *i.i.d.*, $\omega^2 = \sigma^2$, the nuisance parameters vanish and the limiting distributions coincide with those given in (7) and (8).

Various approaches have been suggested in the literature to account for the presence of nuisance parameters in the limiting distributions of $T\left(\widehat{\alpha} - 1\right)$ and t_α in (10) and (11). It was shown by Dickey and Fuller (1979) that when u_t is a finite order AR process of order k, then $T\left(\widehat{\alpha} - 1\right)$ and t_α (known as the augmented Dickey-Fuller tests) based on the regression

$$y_t = \alpha y_{t-1} + \sum_{j=1}^{k-1} \gamma_j \Delta y_{t-j} + v_{tk} \qquad (t = k+1, \ldots, T) \tag{12}$$

have the asymptotic distributions (7) and (8). However, this result does not apply to more general processes when u_t is an ARMA(p, q) process (with $q \geq 1$). In this case a fixed truncation of the augmented Dickey-Fuller regression (12) with $k = \infty$ provides an inadequate solution to the nuisance parameter problem. Following results of Said and Dickey (1984) it has been shown by Chang and Park (2002), however, that when u_t follows an ARMA(p, q) process, then the limiting null distributions of $T\left(\widehat{\alpha} - 1\right)$ and t_α coincide with the nuisance parameter free Dickey-Fuller distributions, provided that ε_t has a finite fourth moment and k increases with the sample such that $k = o\left(T^{1/2-\delta}\right)$ for some $\delta > 0$.

It has been documented in numerous studies, see e.g. Schwert (1989) and Agiakloglou and Newbold (1992), that the augmented Dickey-Fuller tests suffer from size distortion in finite samples in the presence of serial correlation, especially when the dependence is of (negative) moving average type. Ng and Perron (1995, 2001) have further scrutinized rules for truncating long autoregressions when performing unit root tests based on (12). Consider the information criterion

$$IC\left(k\right) = \log \tilde{\sigma}_k^2 + kC_T/T, \qquad \tilde{\sigma}_k^2 = \left(T-k\right)^{-1} \sum_{t=k+1}^{T} \tilde{v}_{tk}^2. \tag{13}$$

Here $\{C_T\}$ is a positive sequence satisfying $C_T = o(T)$. The Akaike Information Criterion (AIC) uses $C_T = 2$, whereas the Schwarz or Bayesian Information Criterion (BIC) sets $C_T = \log T$. Ng and Perron (1995) find that generally these criteria select a too low value of k, which is a source for size distortion. They also show that by using a sequential data dependent procedure, where the significance of coefficients of additional lags is sequentially tested, one obtains a test with improved size. This procedure, however, often leads to overparametrization and power losses. An information criterion designed for integrated time series which alleviates these problems has been suggested by Ng and Perron (2001). Their idea is to select some lag order k in the interval between 0 and a preselected value $k_{\max}$, where the upper bound $k_{\max}$ satisfies $k_{\max} = o(T)$. As a practical rule, Ng and Perron (2001) suggest that $k_{\max}$ =int$(12(T/100)^{1/4})$. Their modified form of the AIC is given by

$$MAIC(k) = \log \check{\sigma}_k^2 + 2(\tau_T(k) + k)/(T - k_{\max}), \tag{14}$$

where $\check{\sigma}_k^2 = (T - k_{\max})^{-1} \sum_{t=k_{\max}+1}^{T} \tilde{v}_{tk}^2$ and $\tau_T(k) = \check{\sigma}_k^{-2}(\tilde{\rho} - 1)^2 \sum_{t=k_{\max}+1}^{T} y_{t-1}^2$. Note that the penalty function is data dependent which captures the property that the bias in the sum of the autoregressive coefficients (i.e., $\widehat{\alpha} - 1$) is highly dependent upon the selected truncation k. Ng and Perron have documented that the modified information criterion is superior to conventional information criteria in truncating long autoregressions with integrated variables when (negative) moving average errors are present.

3.3 Semi-parametric Z tests

Instead of solving the nuisance parameter problem parametrically as in the augmented Dickey-Fuller test, Phillips (1987) and Phillips and Perron (1988) suggest to transform the statistics $T(\widehat{\alpha} - 1)$ and $t_{\widehat{\alpha}}$ based on estimating the model (6) in such a way that the influence of nuisance parameters is eliminated asymptotically. This can be done after consistent estimates of the nuisance parameters ω^2 and σ^2 have been found. More specifically, they suggest the statistics

$$Z_\alpha = T(\widehat{\alpha} - 1) - \frac{\hat{\omega}^2 - \hat{\sigma}^2}{2T^{-2} \sum_{t=1}^{T} y_{t-1}^2} \tag{15}$$

and

$$Z_{t_\alpha} = \frac{\hat{\sigma}}{\hat{\omega}} t_{\widehat{\alpha}} - \frac{\hat{\omega}^2 - \hat{\sigma}^2}{2\sqrt{\hat{\omega}^2 T^{-2} \sum_{t=1}^{T} y_{t-1}^2}}. \tag{16}$$

The limiting null distributions of Phillips' and Perron's Z statistics correspond to the pivotal distributions (7) and (8).

A consistent estimate of σ^2 is

$$\hat{\sigma}^2 = T^{-1} \sum_{t=1}^{T} \hat{u}_t^2, \qquad \widehat{u}_t = y_t - \widehat{\alpha} y_{t-1},$$

whereas for the estimate of the long run variance ω^2 a range of kernel estimators can been considered. These are typically estimators used in spectral density estimation and are of the form

$$\hat{\omega}^2_{KER} = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2\sum_{j=1}^{T-1} w\left(j/M_T\right)\left(T^{-1}\sum_{t=j+1}^{T}\hat{u}_t\hat{u}_{t-j}\right), \tag{17}$$

where $w(\cdot)$ is a kernel (weight) function and M_T is a bandwidth parameter, see e.g. Newey and West (1987) and Andrews (1991).

Even though kernel estimators of the long run variance ω^2 such as (17) are commonly used to remove the influence of nuisance parameters in the asymptotic distributions it has been shown by Perron and Ng (1996) that spectral density estimators cannot generally eliminate size distortions. In fact, kernel based estimators tend to aggravate the size distortions, which is also documented in many Monte Carlo studies, e.g. Schwert (1989), Agiakloglou and Newbold (1992) and Kim and Schmidt (1990). The size distortions arise because the estimation of α and ω^2 are coupled in the sense that the least squares estimator $\widehat{\alpha}$ is used in constructing $\hat{u}_t$ and hence affects $\hat{\omega}^2_{KER}$. The finite (and even large) sample bias of $\widehat{\alpha}$ is well known when u_t exhibits strong serial dependence, and hence the nuisance parameter estimator $\hat{\omega}^2_{KER}$ is expected to be very imprecise.

Following work by Berk (1974) and Stock (1999), Perron and Ng (1996) have suggested a consistent autoregressive spectral density estimator which is less affected by the dependence on $\widehat{\alpha}$. The estimator is based on estimation of the long autoregression (12):

$$\hat{\omega}^2_{AR} = \frac{\tilde{\sigma}^2_k}{\left(1-\sum_{j=1}^{k-1}\tilde{\gamma}_j\right)^2}, \tag{18}$$

where k is chosen according to the information criterion (14). The filtered estimator (18) decouples estimation of ω^2 from the estimation of α and is therefore unaffected by any bias that $\widehat{\alpha}$ may otherwise have due the presence of serial correlation.

3.4 The M Class of Unit Root Tests

When comparing the size properties of the Phillips-Perron tests using the estimator $\hat{\omega}^2_{AR}$ and the tests applying the commonly used Bartlett kernel estimator of ω^2 with linearly decaying weights, Perron and Ng (1996) found significant size improvements in the most critical parameter space. Notwithstanding, size distortions can still be severe and remain so even if $\hat{\omega}^2_{AR}$ is replaced by the (unknown) true value ω^2. This suggests that the bias of $\widehat{\alpha}$ is itself an important source of the size distortions. With this motivation Perron and Ng (1996) and Ng and Perron (2001) suggest further improvements of existing tests with much better size behaviour compared to other tests. Moreover, the tests can be designed such that they satisfy desirable optimality criteria.

The trinity of M tests belongs to a class of tests which has been originally suggested by Stock (1999). They build on the Z class of semiparametric tests but are modified in a particular way to deal with the bias of $\widehat{\alpha}$ and exploit the fact that a series converges at different rates under the null and the alternative hypothesis. The first statistic can be formulated as

$$MZ_\alpha = Z_\alpha + \frac{T}{2}\left(\widehat{\alpha} - 1\right)^2. \tag{19}$$

Since the least squares estimator $\widehat{\alpha}$ is super-consistent under the null, i.e. $\widehat{\alpha} - 1 = O_p\left(T^{-1}\right)$, it follows that Z_α and MZ_α have the same asymptotic distribution. In particular, this implies that the limiting null distribution of MZ_α is the one given in (7). The next M statistic reads

$$MSB = \sqrt{\hat{\omega}_{AR}^{-2} T^{-2} \sum_{t=2}^{T} y_{t-1}^2}, \tag{20}$$

which is stochastically bounded under the null and $O_p\left(T^{-1}\right)$ under the alternative, see also Sargan and Bhargava (1983) and Stock (1999). Note that $Z_{t_\alpha} = MSB \cdot Z_\alpha$, and hence a modified Phillips-Perron t test can be constructed as

$$MZ_{t_\alpha} = Z_{t_\alpha} + \frac{1}{2}\left(\sqrt{\hat{\omega}_{AR}^{-2} \sum_{t=2}^{T} y_{t-1}^2}\right)\left(\widehat{\alpha} - 1\right)^2. \tag{21}$$

The correction factors of MZ_α and MZ_{t_α} can be significant despite the super-consistency of $\widehat{\alpha}$. Perron and Ng show that the M-tests have lower size distortion relative to competing unit root tests. The success of the test is mainly due to the use of the sieve autoregressive spectral density estimator $\hat{\omega}_{AR}^2$ in (18) as an estimator of ω^2. Interestingly, the M-tests also happen to be robust to e.g. measurement errors and additive outliers in the observed series, see Vogelsang (1999).

4. Unit root testing with deterministic components but no breaks

Since many macroeconomic time series are likely to have some kind of deterministic component, it is commonplace to apply unit root tests that yield inference which is invariant to the presence of a particular deterministic component. In practice, a constant term is always included in the model so the concern in most cases is that of whether to include or not to include a linear trend in the model. In many cases auxiliary information may be useful in ruling out a linear trend, for instance for interest rate data, real exchange rates, or inflation rates. However, for many other time series a linear trend is certainly a possibility such as GDP per capita, industrial production, and consumer prices (in logs).

In the previous section we excluded deterministic components from the analysis. Now we consider the model (1) in the special case where

$$f(t) = d_t' \mu. \tag{22}$$

In (22) d_t is a k-vector of deterministic terms, ($k \geq 1$), and μ is a parameter vector of matching dimension. Hence the trend considered is linear-in-parameters. In particular, we will consider in this section the cases where $d_t = 1$ or $d_t = (1, t)'$ since these are the most relevant situations in applications. In principle, however, the analysis can even be extended to higher-order polynomial trends. Consequently we assume the possibility of a level effect or a level plus trend effect (without breaks) in the model. In fact, the linear-in-parameters specification also includes structural breaks of the additive outlier form discussed in section 2.1 when the break date is known. We return to this case later.

4.1 Linear-in-parameters trends without breaks

When allowing for deterministic components the augmented Dickey-Fuller (or Said-Dickey) regressions should take the alternative form

$$y_t = d_t'\mu + \alpha y_{t-1} + \sum_{j=1}^{k-1} \gamma_j \Delta y_{t-j} + v_{tk}. \tag{23}$$

In a similar fashion the Phillips-Perron Z tests allow inclusion of deterministic components in the auxiliary regressions, but alternatively one can also detrend the series prior to testing for a unit root. In all cases where the models are augmented by deterministic components the relevant distributions change accordingly; Brownian motion processes should be replaced with demeaned and detrended Brownian motions of the form

$$W^d(r) = W(r) - D(r)' \left(\int_0^1 D(s) D(s)' ds \right)^{-1} \left(\int_0^1 D(s) W(s) ds \right), \tag{24}$$

where $D(r) = 1$ when $d_t = 1$ and $D(r) = (1, r)'$ when $d_t = (1, t)'$. The relevant distributions of the ADF and Phillips-Perron tests are reported in Fuller (1996).

Concerning the M-tests discussed in Section 3.4, Perron and Ng (2001) suggest an alternative way to treat deterministic terms. They recommend to adopt the local GLS detrending/demeaning procedure initially developed by Elliott, Rothenberg and Stock (1996). This has the further advantage that tests are "nearly" efficient in the sense that they nearly achieve the asymptotic power envelopes for unit root tests. For a time series $\{x_t\}_{t=1}^T$ of length T and any constant $\bar{c}$, define the vector $x^{\bar{c}} = (x_1, \Delta x_2 - \bar{c}T^{-1}x_1, \ldots, \Delta x_T - \bar{c}T^{-1}x_{T-1})'$. The so-called GLS detrended series $\{\tilde{y}_t\}$ reads

$$\tilde{y}_t = y_t - d_t'\tilde{\beta}, \qquad \tilde{\beta} = \arg\min_{\beta} \left(y^{\bar{c}} - d^{\bar{c}\prime}\beta\right)' \left(y^{\bar{c}} - d^{\bar{c}\prime}\beta\right). \tag{25}$$

Elliott, Rothenberg and Stock (1996) suggested $\bar{c} = -7$ and $\bar{c} = -13.5$ for $d_t = 1$ and $d_t = (1, t)'$, respectively. These values of $\bar{c}$ correspond to the local alternatives against which the local asymptotic power envelope for 5% tests equals 50%. When the M-tests are constructed by using GLS detrended data and the long run variance estimator is $\hat{\omega}^2_{AR}$ defined in (18) together with the modified information criterion (14), the tests are denoted MZ_α^{GLS}, MSB^{GLS}, and MZ_t^{GLS}, respectively. Ng and Perron (2001) show that these tests have both excellent size and power when a moving average component is present in the error process.

The efficient tests suggested by Elliott, Rothenberg and Stock (1996) have become a benchmark reference in the literature. A class of these tests which we will later refer to is the GLS detrended or demeaned ADF tests, ADF_t^{GLS}. Basically these tests are constructed from the Dickey-Fuller t-statistic based on an augmented Dickey-Fuller regression without deterministic terms, like (12), where $\widetilde{y}_t$ in (25) is used in place of y_t. The lag truncation again needs to be chosen via a consistent model selection procedure like the $MAIC$ criterion in (14). These tests can be shown to be near asymptotically efficient. The GLS detrended or demeaned ADF and M-tests can thus be considered parametric and semi-nonparametric tests designed to be nearly efficient. Regarding the $t-$statistic based tests ADF_t^{GLS} and MZ_t^{GLS}, the distributions for $d_t = 1$ is the Dickey-Fuller distribution for the case without deterministics (8) for which the critical values are reported in Fuller (1996). When $d_t = (1, t)'$ the relevant distribution can be found in Elliott, Rothenberg and Stock (1996) and Ng and Perron (2001) where the critical values for $\bar{c} = -13.5$ are also reported.

As a final note, Perron and Qu (2007) have shown that the selection of k using MAIC with GLS demeaned or detrended data may lead to power reversal problems, meaning that the power against non-local alternatives may be small. However, they propose a simple solution to this problem; first select k using the MAIC with OLS demeaned or detrended data and then use this optimal autoregressive order for the GLS detrended test.

4.2 Uncertainty about the trend

The correct specification of deterministic components is of utmost importance to conduct consistent and efficient inference of the unit root hypothesis. Regarding the Dickey-Fuller class of tests one would always allow for a nonzero mean at least. However, it is not always obvious whether one should also allow for a linear trend. A general (conservative) advice is that if it is desirable to gain power against the trend-stationary alternative, then a trend should be included in the auxiliary regression (23), i.e., $d_t = (1, t)'$. Note that under the null hypothesis the expression (5) shows that the coefficient of the trend regressor is zero in the model (23). However, if one does not include a linear time trend in (23) it will be impossible to have power against the trend-stationary alternative, i.e. asymptotic power will be trivial, leading to the conclusion that a unit root is present when the alternative is in fact true. In other words, the distribution of the unit root test statistic allowing for an intercept but no trend is not invariant to the actual value of the trend.

Although the use of a test statistic which is invariant to trends seems favourable in this light, it turns out that the costs in terms of power loss can be rather significant when there is no trend, see Harvey, Leybourne, and Taylor (2009). In fact, this is a general finding for unit root tests where the decision of demeaning or detrending is an integral part of the testing procedure. Therefore, it would be of great value if some prior knowledge were available regarding the presence of a linear trend, but generally such information does not exist.

A possible strategy is to make pre-testing for a linear trend in the data an integral part of the unit root testing problem, i.e. the unit root test should be made contingent on the outcome of the trend pre-testing. However, this is complicated by the fact that we do not know whether the time series is I(1) or I(0) (this is what is being tested), and in the former case spurious evidence of trends may easily occur. A rather large literature exists on trend testing where test statistics are robust to the integration order of the data; a non-exhaustive list of references includes Phillips and Durlauf (1988), Vogelsang (1998), Bunzel and Vogelsang (2005), and Harvey, Leybourne and Taylor (2007). It turns out that testing for a trend prior to unit root testing generally is statistically inferior to alternative approaches.

Harvey, Leybourne and Taylor (2009) discuss a number of different approaches to dealing with the uncertainty about the trend. In particular, they consider a strategy which entails *pre-testing* of the trend specification, a strategy based on a *weighting scheme* of the Elliott, Rothenberg and Stock (1996) GLS demeaned or detrended ADF unit root tests, and finally a strategy based on a *union of rejections* of the GLS demeaned and detrended ADF tests. They find that the last procedure is better than the first two. It is also straightforward to apply since it does not require an explicit form of trend detection via an auxiliary statistic. Moreover, the strategy has practical relevance since it embodies what applied researchers already do, though in an implicit manner. Even though it is beyond the study of Harvey, Leybourne and Taylor (2009) our conjecture is that the strategy can be equally applied to the GLS filtered MZ_t^{GLS}-tests.

The idea behind the union of rejections strategy is a simple decision rule stating that one should reject the $I(1)$ null if either $ADF_t^{GLS,1}$ (demeaning case) or $ADF_t^{GLS,(1,t)}$ (detrending case) rejects. The union of rejection test is defined as

$$UR = ADF_t^{GLS,1}\mathbf{I}(ADF_t^{GLS,1} < -1.94) + ADF_t^{GLS,(1,t)}\mathbf{I}(ADF_t^{GLS,1} \geq -1.94).$$

Here $\mathbf{I}(.)$ is the indicator function. If $UR = ADF_t^{GLS,1}$ the test rejects when $UR < -1.94$ and otherwise, if $UR = ADF_t^{GLS,(1,t)}$ a rejection is recorded when $UR < -2.85$. Alternatively, the $MZ_t^{GLS,1}$ and $MZ_t^{GLS,(1,t)}$ tests can be used in place of the ADF_t^{GLS} tests; the asymptotic distributions will be the same. Based on the asymptotic performance and finite sample analysis Harvey, Leybourne and Taylor (2009) conclude that despite its simplicity the UR test offers very robust overall performance compared to competing strategies and is thus useful for practical applications.

5. The initial condition

So far we have assumed that the initial condition $y_0 = 0$. This may seem to be a rather innocuous requirement since it can be shown that as long as $y_0 = o_p(T^{1/2})$ the impact of the initial observations is asymptotically negligible. For instance, this is the case when the initial condition is modelled as a constant nuisance parameter or as a random variable with a known distribution. However, problems occur under the alternative hypothesis and hence have implications to the power of unit root tests.

To clarify the argument, assume that y_t follows a stationary Gaussian AR(1) process with no deterministic components and let the autoregressive parameter be α. In this situation $y_0 \sim N(0, 1/(1-\alpha^2))$. If we assume a local discrepancy from the unit root model, i.e. by defining $\alpha = 1 + c/T$ to be a parameter local to unity where $c < 0$, then it follows that $T^{-1/2}y_0 \sim N(0, -1/(2c))$. For the initial value to be asymptotically negligible we thus require this to be of a smaller order in probability than the remaining data points which seems to be an odd property. Since stationarity is the reasonable alternative when testing for a unit root, this example shows that even for a rather simple model the impact of the initial observation is worth examining.

In practical situations it is hard to rule out small or large initial values a priori. As noted by Elliott and Müller (2006) there may be situations where one would not expect the initial condition to be exceptionally large or small relative to other observations. Müller and Elliott (2003) show that the influence of the initial condition can be rather severe in terms of power of unit root tests and the fact that what we observe is the initial observation, not the initial condition, see Elliott and Müller (2006). In practice this means that different conclusions can be reached with samples of the same data which only differ by the date at which the time series begins. Müller and Elliott (2003) derive a family of efficient tests which allow attaching different weighting schemes to the initial condition. They explore the extent to which existing unit root tests belong to this class of optimal tests. In particular they show that certain versions of the Dickey-Fuller class of tests are well approximated by members of efficient tests even though a complete removal of the initial observation influence cannot be obtained.

Harvey, Leybourne and Taylor (2009) undertake a very detailed study of strategies for dealing with uncertainty about the initial condition. The study embeds the set up in the papers by Elliott and Müller as special cases. Interestingly, Harvey *et al.* find that when the initial condition is not negligible asymptotically the ADF_t^{GLS} class of tests of Elliott, Rothenberg and Stock (1996) can perform extremely poorly in terms of asymptotic power which tends to zero as the magnitude of the initial value increases. This contrasts the performance of the ADF_t^{GLS} tests when the initial condition is "small". However, the usual ADF tests using demeaned or detrended data from OLS regressions have power that increases with the initial observation, and hence are preferable when the initial value is "large". This finding made Harvey *et al.* suggest a union of rejections based test similar to the approach adopted when there is uncertainty about the presence of a trend in the data. Their rule is to reject the unit root null if either the detrended (demeaned) ADF_t^{GLS} and the OLS detrended (demeaned) ADF test rejects. Their study shows that in terms of both asymptotic and finite sample behaviour the suggested procedure has properties similar to the optimal tests suggested by Elliott and Müller (2006). This means that despite its simplicity the procedure is extremely useful as a practical device for unit root testing.

In practical situations there will typically be uncertainty about both the initial condition and the deterministic components when testing for a unit root. In a recent paper Harvey, Leybourne, and Taylor (2012) provide a joint treatment of these two major problems based on the approach suggested in their previous works.

6. Unit roots and structural breaks

In section 2.1 it was argued that when structural breaks are present in the time series, they share features similar to unit root processes. This is most apparent when analyzing the statistical properties of unit root tests in the presence of breaks. It follows from the work of Perron (1989, 1990) that in such circumstances inference can be strongly misleading. For instance, a deterministic level shift will cause $\widehat{\alpha}$ from the augmented Dickey-Fuller regression to be biased towards 1 and a change in the trend slope makes the estimator tend to 1 in probability as the sample size increases. Thus, the DF test will indicate the presence of a unit root even when the time series is stationary around the deterministic break component. In fact, these problems concern most unit root tests including the tests belonging to the Phillips-Perron class of tests, see Montañes and Reyes (1998, 1999). A practical concern therefore is how to construct appropriate testing procedures for a unit root when breaks occur and power against the break alternative is wanted. In practice, what is important is to identify any major breaks in the data since these would otherwise give rise to the largest power loss of unit root tests. Minor breaks are more difficult to detect in finite samples than large ones, but then they only lead to minor power reductions. Hence the importance of focusing on "large" breaks.

6.1 Unit root testing accounting for a break at known date

Additive outlier breaks. The way unit root tests should be formulated under the break hypothesis depends on the type of break considered. Also, it is crucial for the construction of tests that both the null and alternative is permitted in the model specification with the autoregressive root allowed to vary freely. Here we present variants of the Dickey-Fuller tests where the breaks allowed for are those of Perron (1989, 1990) with a known single break date. This corresponds to the models discussed in section 2.1.

First we consider breaks belonging to the additive outlier class. In this situation the testing procedure relies on two steps. In the first step the series y_t is detrended, and in the second step an appropriately formulated augmented Dickey-Fuller test with additional dummy regressors included is applied to the detrended series. For the additive outlier models $AO^a - AO^d$ the detrended series are obtained by regressing y_t on all the relevant deterministic terms that characterize the model. That is, the detrended series in the additive outlier model with breaks in both the level and the trend, (model AO^d), is constructed as

$$y_t = \widetilde{\mu}_1 + \widetilde{\beta}t + \widetilde{\mu}_1^* DU_t + \widetilde{\beta}^* DT_t + \widetilde{y}_t \qquad (26)$$

where the parameters are estimated by the least squares method and $\widetilde{y}_t$ is the detrended series. The detrended series for the other AO models are nested within the above detrending regression by appropriate exclusion of irrelevant regressors for the particular model considered. The unit root null is tested via a Dickey-Fuller $t-$test of $\widetilde{y}_t$. For the additive outlier model (AO^c) (with DU_t absent from (26)) the Dickey-Fuller regression is like (12) applied to $\widetilde{y}_t$. For the remaining AO models which all include DU_t as a deterministic component, the auxiliary regression takes the form

$$\widetilde{y}_t = \widehat{\alpha}\widetilde{y}_{t-1} + \sum_{j=1}^{k-1} \widehat{d}_j (1-L) DU_{t-j} + \sum_{j=1}^{k-1} \widehat{c}_j \Delta \widetilde{y}_{t-j} + e_t \qquad (t = k+1, \ldots, T) \qquad (27)$$

where k can be selected according to the criteria previously discussed. Note however, that the selection of k using the $MAIC$ criterion can be affected by the presence of breaks and hence other information criteria may by considered as well, see Ng and Perron (2001), Theorem 3. Observe that the inclusion of consecutive impulse dummy variables $(1-L)DU_{t-j}$ is a temporary level shift patch that is caused by the general dynamics of the model. From this it can also be seen why this component is absent in the AO^c model. It can be shown that the null distribution of the Dickey-Fuller t-test from this regression will depend upon the actual timing of the break date T_1. If λ_1 determines the in-sample fraction of the full sample where the break occurs, i.e. $\lambda_1 = T_1/T$, then the distribution reads

$$t_\alpha(\lambda_1) \xrightarrow{d} \frac{\int_0^1 W^d(r, \lambda_1) dW(r)}{\sqrt{\int_0^1 W^d(r, \lambda_1)^2 dr}} \qquad (28)$$

where $W^d(r, \lambda_1)$ is a process which is the residual function from a projection of $W(r)$ on the relevant continous time equivalent of the deterministic components used to detrend y_t, i.e.

1, $I(r > r_1), r, I(r > \lambda_1)(r - \lambda_1)$ depending on the model. The relevant distributions (28) are tabulated in Perron (1989, 1990). Critical values for the AO^c case are reported in Perron and Vogelsang (1993).

Innovation outlier breaks. The innovation outlier models under both the null and alternative hypotheses can all be encompassed in the model

$$y_t = \mu + \theta DU_t + \beta t + \gamma DT_t + \delta(1 - L)DU_t + \alpha y_{t-1} + \sum_{j=1}^{k-1} c_i \Delta y_{t-j} + e_t. \qquad (29)$$

The regressors t and DT_t are absent from the IO^a. The IO^b model does not contain DT_t. Under the null hypothesis $\alpha = 1$ and for a components representation this would generally imply that many of the coefficients of the deterministic components would equal zero, even though these restrictions are typically not imposed when formulating tests. Note however, that when there is in fact a level shift under the null, then $\delta \neq 0$ whereas under the alternative $\delta = 0$ and the remaining coefficients will typically be nonzero. By construction, the Dickey-Fuller t-statistic from this regression is invariant to the mean and trend as well as a possible break in them, provided the break date is correct. The distribution of $t_{\widehat{\alpha}}(\lambda_1)$ is in this case identical to that given in (28).

The tests of Perron with known break dates have been generalized and extended in a number of directions. Saikkonen and Lütkepohl (2002) consider a class of the AO type of models that allow for a level shift whereas Lütkepohl, Müller and Saikkonen (2001) consider level shift models of the IO type. They propose a GLS-type detrending procedure and a unit root test statistic which has a limiting null distribution that does not depend upon the break date. Invariance with respect to the break date is a result of GLS detrending and in fact the relevant distribution is that of Elliott *et al.* (1996) for the case with a constant term included as a regressor in their model. This approach has been shown, see Lanne and Lütkepohl (2002), to have better size and power than the test proposed by Perron (1990).

It is important to stress, however, that there are many ways of misspecifying break dates and choosing an incorrect break model will affect inference negatively. The dates of possible breaks are usually unknown unless they refer to particular historical or economic events. Hence procedures for unit root testing when the break date is unknown are necessary. Such procedures will be discussed next.

6.2 Unit root testing accounting for break at unknown date

For the tests of Perron (1989, 1990) to be valid, the break date should be chosen independently of the given data and it has been argued by Christiano (1992) for instance that treating the break date as fixed in many cases is inappropriate. In practical situations the search and identification of breaks implies pretesting that will distort tests that use critical values for known break date unit root tests. Of course this criticism is only valid if in fact a search has been conducted to find the breaks. On the other hand, such a procedure despite of having a correct size may result in power loss if the break date is given without pretesting. When the break date is unknown Banerjee, Lumsdaine and Stock (1992) suggested to consider a sequence of rolling or recursive tests and then use the minimal value of the unit root test and reject the null if the test value is sufficiently small. However, because such procedures will be based on sub-sampling it is expected (and proven in simulations) that finite sample power loss will result.

Zivot and Andrews (1992) suggested a procedure which in some respects is closely linked to the methodology of Perron (1989) whereas in other respects it is somewhat different. Their model is of the IO type. For the IO^d model they consider the auxiliary regression model

$$y_t = \mu + \theta DU_t + \beta t + \gamma DT_t + \alpha y_{t-1} + \sum_{j=1}^{k-1} c_i \Delta y_{t-j} + e_t \tag{30}$$

which essentially is (29) leaving out $(1-L)DU_t$ as a regressor. The test that $\alpha = 1$ is based on the minimal value of the associated $t-$ratio, $t_\alpha(\lambda_1)$, over all possible break dates in some range of the break fraction that is pre-specified, i.e. $[\varepsilon, 1-\varepsilon]$. Typically one sets $\varepsilon = 0.15$ even though Perron (1997) has shown that trimming the break dates is unnecessary. The resulting test statistic has the distribution

$$t_\alpha^* = \inf_{\lambda_1 \in [\varepsilon, 1-\varepsilon]} t_\alpha(\lambda_1) \xrightarrow{d} \inf_{\lambda_1 \in [\varepsilon, 1-\varepsilon]} \frac{\int_0^1 W^d(r, \lambda_1) dW(r)}{\sqrt{\int_0^1 W^d(r, \lambda_1)^2 dr}} \tag{31}$$

with $W^d(r, \lambda_1)$ defined in section 6.1.

The analysis of Andrews and Zivot (1992) has been extended by Perron and Vogelsang (1992) and Perron (1997) for the non-trending and trending cases, respectively. They consider both the IO and AO models based on the appropriately defined minimal value $t-$statistic of the null hypothesis. They also consider using the test statistics $t_\alpha(\lambda_1)$ for the case of known break date where the break date T_1 is determined by maximizing the numerically largest value of the $t-$statistic associated with the coefficient of the shift dummy DU_t (in case of a level shift) and DT_t (in case of slope change). Regarding the IO model they also consider using the regression (29) rather than (30) since this would be the right regression with a known break date.

A problem with the tests that build upon the procedure of Zivot and Andrews (1992) is that a break is not allowed under the null hypothesis of a unit root but only under the alternative. Hence the deterministic components are given an asymmetric treatment under the null and alternative hypotheses. This is a very undesirable feature, and Vogelsang and Perron (1998) showed that if a unit root exists and a break occurs in the trend function the Zivot and Andrews test will either diverge or will not be invariant to the break parameters. This caveat has motivated Kim and Perron (2009) to suggest a test procedure which allows a break in the trend function at unknown date under both the null and alternative hypotheses. The procedure has the advantage that when a break is present, the limiting distribution of the test is the same as when the break date is known, which increases the power whilst maintaining the correct size.

Basically, Kim and Perron (2009) consider the class of models initially suggested by Perron (1989) with the modification that a possible break date T_1 is assumed to be unknown. The models they address are the additive outlier models that allow for a non-zero trend slope AO^b, AO^c and AO^d, and the innovation outlier models associated with IO^b and IO^d, that is, the models implying a level shift, changing growth, or a combination of the two.

The IO^c models are not considered since it is necessary to assume that no break occurs under the null hypothesis which contradicts the purpose of the analysis. When T_1 is an unknown parameter it is difficult in practice to estimate the models because the form of the regressors to be included is unknown. Notwithstanding, an estimate of the break date may be considered. The idea is to consider conditions under which the distribution of $t_\alpha(\widehat{\lambda}_1)$ for the additive outlier case, for instance, is the same as the distribution of $t_\alpha(\lambda_1)$ given in (28); in other words, the limiting distribution of the Perron test is unaffected of whether the break is known in advance or has been estimated and hence the critical values for the known break date can be used. It turns out that such a result will depend upon the consistency rate of the estimate of the break fraction and also whether or not there is a non-zero break occuring in the trend slope of the model. Suppose that we have a consistent estimate $\widehat{\lambda}_1 = \widehat{T}_1/T$ of the break fraction such that

$$\widehat{\lambda}_1 - \lambda_1 = O_p(T^{-a}) \tag{32}$$

for some $a \geq 0$. For the models AO^c, AO^d, and IO^d with a non-zero break the distribution of the unit root hypothesis for the estimated break date case is the same as the case of known break date when $\widehat{\lambda}_1 - \lambda_1 = o_p(T^{-1/2})$. A consistent estimate of the break date is therefore not needed, but the break fraction needs to be consistently estimated at a rate larger than $\sqrt{T}$.

Kim and Perron (2009) consider a range of different estimators that can be used to estimate the break fraction with different rates of consistency. They also suggest an estimator using trimmed data whereby the rate of convergence can be increased. The reader is referred to Kim and Perron's (2009) paper for details.

So far we have assumed that a break in trend occurs under both the null and alternative hypotheses. When there is no such break the asymptotic results will no longer hold because the estimate of the break fraction will have a non-generate distribution on [0,1] under the null. Hence a pre-testing procedure is needed to check for a break, see e.g. Kim and Perron (2009), Perron and Yabu (2009a), Perron and Zhu (2005), and Harris *et al.* (2009). When the outcome of such a test indicates that there is no trend break the usual Dickey-Fuller class of tests can be used.

In the models with a level shift, i.e. models AO^b and IO^b, Kim and Perron (2009) show that known break date asymptotics will apply as long as the break fraction is consistently estimated. On the other hand, if the break happens to be large in the sense that the magnitude of the break increases with the sample size, then a consistent estimate of the break fraction is not enough and a consistent estimate of the break date itself is needed.

Our review of unit root tests with unknown break dates is necessarily selective. Contributions not discussed here include Perron and Rodríguez (2003), Perron and Zhu (2005), and Harris *et al.* (2009) among others. Harris *et al.* (2009) suggests a procedure that is adequate when there is uncertainty about what breaks occur in the data. In so doing they generalize the approaches discussed in sections 4.2 and 5 where there is uncertainty about the trend or the initial condition.

There is also a large literature dealing with the possibility of *multiple* breaks in time series that are either known or unknown. We will not discuss these contributions but simply note that even though accounting for multiple breaks may give testing procedures with a controllable size, the cost will typically be a power loss that can often be rather significant, see Perron (2006) for a review.

7. Unit root testing against nonlinear models

This section is dedicated to recent developments in the field of unit root tests against stationary nonlinear models. Amongst the nonlinear models we consider (i) smooth transition autoregressive (STAR) models and (ii) threshold autoregressive (TAR) models. The model under validity of the null hypothesis is linear which is in conjunction with the majority of the literature. A notable exception is Yildirim, Becker and Osborn (2009) who suggest to consider nonlinear models under both, H_0 and H_1.

When smooth transition models are considered as an alternative to linear non-stationary models one often finds the Exponential STAR (ESTAR) model to be the most popular one; see, however, Eklund (2003) who considers the logistic version of the STAR model under the alternative. In particular, a three-regime ESTAR specification is used in most cases. The middle regime is nonstationary, while the two outer regimes are symmetric and mean-reverting. A prototypical model specification is given by

$$\Delta y_t = \rho y_{t-1}(1 - \exp(-\gamma y_{t-1}^2)) + \varepsilon_t \ , \gamma > 0, \rho < 0 \ ,$$

see Haggan and Ozaki (1981). Kapetanios, Shin and Snell (2003) prove geometric ergodicity of such a model. A problem which is common to all linearity tests against smooth transition models (including TAR and Markov Switching (MS) models) is the Davies (1987) problem. Usually, at least one parameter is not identified under the null hypothesis, see Teräsvirta (1994). As unit root tests also take linearity as part of the null hypothesis, this problem becomes relevant here as well. The shape parameter γ is unidentified under $H_0 : \rho = 0$. As shown by Luukkonen, Saikkonen and Teräsvirta (1988), this problem can be circumvented by using a Taylor approximation of the nonlinear transition function $(1 - \exp(-\gamma y_{t-1}^2))$ around $\gamma = 0$. The resulting auxiliary test regression is very similar to a standard Dickey-Fuller test regression, see Kapetanios *et al.* (2003):

$$\Delta y_t = \delta y_{t-1}^3 + u_t \ .$$

The limiting distribution of the t-statistic for the null hypothesis of $\delta = 0$ is nonstandard and depends on functionals of Brownian motions. The popularity of the test by Kapetanios *et al.* (2003) may stem from its ease of application. Regarding serially correlated errors, Kapetanios *et al.* (2003) suggest augmenting the test regression by lagged differences, while Rothe and Sibbertsen (2006) consider a Phillips-Perron-type adjustment. Another issue is tackled in Kruse (2011) who proposes a modification of the test by allowing for non-zero location parameter in the transition function. Park and Shintani (2005) and Kilic (2011) suggest to deal with the Davies problem in a different way. Rather than applying a Taylor approximation to the transition function, these authors consider an approach which is commonly used in the framework of TAR models, i.e. a grid-search over the unidentified parameters, see also below.

Adjustment of deterministic terms can be handled in a similar way as in the case of linear alternatives. Kapetanios and Shin (2008) suggest GLS-adjustment, while Demetrescu and Kruse (2012) compare also recursive adjustment and the MAX-procedure by Leybourne (1995) in a local-to-unity framework. The findings suggest that GLS-adjustment performs best in the absence of a non-zero initial condition. Similar to the case of linear models, OLS adjustment proves to work best, when the initial condition is more pronounced. Another finding of Demetrescu and Kruse (2012) is that a combination of unit root tests against linear and nonlinear alternatives would be a successful (union-of-rejections) strategy.

The ESTAR model specification discussed above can be viewed as restrictive in the sense that there is only a single point at which the process actually behaves like a random walk, namely at $y_{t-1} = 0$. In situations where it is more reasonable to assume that the middle regime contains multiple points, one may use a double logistic STAR model which is given by

$$\Delta y_t = \rho y_{t-1}(1 + \exp(-\gamma(y_{t-1} - c_1)(y_{t-1} - c_2))^{-1} + \varepsilon_t \ , \gamma > 0, \rho < 0 \ ,$$

see Jansen and Terävirta (1996). Kruse (2011) finds that unit root tests against ESTAR have substantial power against the double logistic STAR alternative although the power is somewhat lower as against ESTAR models. Such a result is due to similar Taylor approximations and suggests that a rejection of the null hypothesis does not necessarily contain information about the specific type of nonlinear adjustment. This issue is further discussed in Kruse, Frömmel, Menkhoff and Sibbertsen (2011).

Another class of persistent nonlinear models which permits a region of nonstationarity is the one of Self-Exciting Threshold Autoregressive (SETAR) models. Regarding unit root tests against three-regime SETAR models, important references are Bec, Salem and Carrasco (2004), Park and Shintani (2005), Kapetanios and Shin (2006) and Bec, Guay and Guerre (2008). Similar to the case of smooth transition models, the middle regime often exhibits a unit root. Importantly, the transition variable is the lagged dependent variable y_{t-1} which is nonstationary under the null hypothesis. In Caner and Hansen (2001), stationary transition variables such as the first difference of the dependent variable are suggested. The TAR model with a unit root in the middle regime is given by (we abstract from intercepts here for simplicity)

$$\Delta y_t = \rho_1 y_{t-1} 1(y_{t-1} \leq -\lambda) + \rho_3 y_{t-1} 1(y_{t-1} \leq \lambda) + \varepsilon_t \ .$$

The nonstationary middle regime with a is defined by $\Delta y_t = \varepsilon_t$ for $|y_{t-1}| < \lambda$. Stationarity and mixing properties of a more general specification of the TAR model are provided in Bec *et al.* (2004). Similar to the STAR model, the parameter λ is not identified under the null hypothesis of a linear unit root process. In order to tackle this problem, sup-type Wald statistics can be considered:

$$\sup W \equiv \sup_{\lambda \in [\underline{\lambda}_T, \bar{\lambda}_T]} W_T(\lambda)$$

A nuisance parameter free limiting distribution of the $\sup LM$ statistic can be achieved by choosing the interval $[\underline{\lambda}_T, \bar{\lambda}_T]$ appropriately. The treatment of the parameter space for λ distinguishes most of the articles. Seo (2008) for example suggests a test based on a compact parameter space similar to Kapetanios and Shin (2006). The test allows for a general dependence structure in the errors and uses the residual-block bootstrap procedure (see Paparoditis and Politis 2003) to calculate asymptotic p-values. Comparative studies are, amongst others, Maki (2009) and Choi and Moh (2007).

8. Unit roots and other special features of the data

We have considered unit root testing for a range of different situations that may characterize economic time series processes. Here we will briefly describe some other features of the data that often occurs to be important for unit root testing. This non-exhaustible review will include issues related to higher order integrated processes, the choice of an appropriate functional form, the presence of heteroscedasticity, and unit root testing when economic variables by their construction are bounded upwards, downwards, or both.

$I(2)$ processes. So far we have addressed the unit root hypothesis implying that the time series of interest is I(1) under the null hypothesis. There has been some focus in the literature on time series processes with double unit roots, so-called I(2) processes. As seen from equation (2) an I(1) process is driven by a stochastic trend component of the form $\sum_{j=1}^{t} \varepsilon_j$. If, on the other hand, u_t in (1) is I(2), then $(1-L)^2 u_t = C(L)\varepsilon_t$, and the series can be shown to include a stochastic trend component of the form

$$\sum_{k=1}^{t}\sum_{j=1}^{k} \varepsilon_j = t\varepsilon_1 + (t-1)\varepsilon_2 + ... + 3\varepsilon_{t-2} + 2\varepsilon_{t-1} + \varepsilon_t. \tag{33}$$

As seen, a shock to the process will have an impact that tends to increase over time. This may seem to be an odd feature but it is implied by the fact that (if the series is log transformed) then shocks to the growth rates are I(1), and hence will persist, and this will further amplify the effect on the level of the series when cumulated. In practice, testing for I(2) is often conducted by testing whether the first differences of the series have a unit root under the null hypothesis. This can be tested using the range of tests available for the I(1) case. See for instance Haldrup (1998) for a review on the statistical analysis of I(2) processes.

Functional form. From practical experience researchers have learned that unit root testing is often sensitive to non-linear transformations of the data. For instance, variables expressed in logarithms are sometimes found to be stationary, whereas the same variables in levels are found to be non-stationary. Granger and Hallman (1991) addressed the issue of appropriate non-linear transformations of the data and developed a test for unit roots that is invariant to monotone transformations of the data such as $y_t^2, y_t^3, |y_t|, sgn(y_t), \sin(y_t)$, and $\exp(y_t)$. Franses and McAleer (1998) developed a test of nonlinear transformation to assess the adequacy of unit root tests of the augmented Dickey-Fuller type. They considered the following generalized augmented Dickey-Fuller regression (ignoring deterministic components) of a possibly unknown transformation of y_t

$$y_t(\delta) = \alpha y_{t-1}(\delta) + \sum_{j=1}^{k-1} \gamma_j \Delta y_{t-j}(\delta) + v_{tk} \qquad (t = k+1, \ldots, T) \tag{34}$$

where $y_t(\delta)$ denotes the transformation of Box and Cox (1964) given by

$$\begin{aligned} y_t(\delta) &= (y_t^{\delta} - 1)/\delta & \delta \neq 0,\ y_t \geq 0 \\ &= \log y_t(\delta) & \delta = 0,\ \ y_t > 0. \end{aligned} \tag{35}$$

For this model Franses and McAleer (1998) considered the null hypothesis of a unit root for some assumed value of the Box-Cox parameter δ, but without estimating the parameter directly. Based on a variable addition test they showed how the adequacy of the transformation could be tested. Fukuda (2006) suggested a procedure based on information criteria to jointly decide on the unit root hypothesis and the transformation parameter.

Bounded time series. Many time series in economics and finance are bounded by construction or are subject to policy control. For instance the unemployment rate and budget shares are variables bounded between zero and one and some variables, exchange rates for instance, may be subject to market intervention within a target zone. Conventional unit root tests will be seriously affected in this situation. Cavaliere (2005) showed that the limiting distributions in this case will depend upon nuisance parameters that reflect the position of the bounds. The tighter bounds, the more will the distribution be shifted towards the left and thus bias the standard tests towards stationarity. Only when the bounds are sufficiently far away will conventional unit root tests behave according to standard asymptotic theory. Cavaliere and Xu (2011) have recently suggested a testing procedure for augmented Dickey-Fuller tests and the autocorrelation-robust $M-$tests of Perron and Ng (1996) and Ng and Perron (2001) even though in principle the procedure can be used for any commonly used test.

The processes considered by Cavaliere and Xu (2011) behave like random walks but are bounded above, below or both; see also Granger (2010). The time series x_t is assumed to have (fixed) bounds at $\underline{b}, \overline{b}$, ($\underline{b} < \overline{b}$), and is a stochastic process $x_t \in [\underline{b}, \overline{b}]$ almost surely for all t. This means that the increments Δx_t necessarily have to lie in the interval $[\underline{b} - x_{t-1}, \overline{b} - x_{t-1}]$. Rewrite the process in the following form

$$\begin{aligned} x_t &= \mu + y_t \\ y_t &= \alpha y_{t-1} + u_t, \alpha = 1 \end{aligned} \tag{36}$$

where u_t is further decomposed as

$$u_t = \varepsilon_t + \underline{\xi}_t - \overline{\xi}_t. \tag{37}$$

Furthermore, ε_t is a weakly dependent zero-mean unbounded process and $\underline{\xi}_t, \overline{\xi}_t$ are non-negative processes satisfying

$$\begin{array}{ll} \underline{\xi}_t > 0 & \text{for } y_{t-1} + \varepsilon_t < \underline{b} - \theta \\ \overline{\xi}_t > 0 & \text{for } y_{t-1} + \varepsilon_t > \overline{b} - \theta. \end{array} \tag{38}$$

A bounded I(1) process x_t will revert because of the bounds. When it is away from the bounds it behaves like a unit root process. When being close to the bounds the presence of the terms $\underline{\xi}_t$ and $\overline{\xi}_t$ will force x_t to lie between $\underline{b}$ and $\overline{b}$. In the stochastic control literature, see Harrison (1985), the stochastic terms $\underline{\xi}_t$and $\overline{\xi}_t$ are referred to as "regulators".

To derive the appropriate asymptotic distributions of the augmented Dickey-Fuller test and the $M-$tests defined in sections 3.2 and 3.4 Cavaliere and Xu (2011) relate the position of the bounds $\underline{b}$ and $\overline{b}$ (relative to the location parameter θ) to the sample size T as $(\underline{b} - \theta)/(\lambda T^{1/2}) = \underline{c} + o(1)$ and $(\overline{b} - \theta)/(\lambda T^{1/2}) = \overline{c} + o(1)$ where $\underline{c} \leq 0 \leq \overline{c}$, $\underline{c} \neq \overline{c}$. It occurs that the parameters $\underline{c}$ and $\overline{c}$ will appear as nuisance parameters in the relevant asymptotic distributions expressed in terms of a regulated Brownian motion $W(r; \underline{c}, \overline{c})$, see Nicolau

(2002). When the bounds are one-sided $\underline{c} = -\infty$ or $\bar{c} = \infty$ and when there are no bounds $W(r; \underline{c}, \bar{c}) \rightarrow W(r)$ for $\underline{c} \rightarrow -\infty$ and $\bar{c} \rightarrow \infty$. Hence the usual bounds-free unit root distributions will apply as a special case.

The lesson to be learned from the analysis is that standard unit root inference is affected in the presence of bounds. If the null hypothesis is rejected on the basis of standard critical values, it is not possible to assess whether this is caused by the absence of a unit root or by the presence of the bounds. On the other hand, the nonrejection of the unit root hypothesis is very strong evidence for the null hypothesis under these circumstances. To provide valid statistical inference, Cavaliere and Xu (2011) suggest a testing procedure based on first estimating the nuisance parameters $\underline{c}$ and $\bar{c}$. Based on these estimates, they suggest simulating the correct asymptotic null distribution from which the asymptotic $p-$value of the unit root test can be inferred. In estimating the nuisance parameters $\underline{c}$ and $\bar{c}$ they define the consistent estimators

$$\widehat{\underline{c}} = \frac{\underline{b} - x_0}{\hat{\omega}^2_{AR} T^{\frac{1}{2}}}, \ \widehat{\bar{c}} = \frac{\bar{b} - x_0}{\hat{\omega}^2_{AR} T^{\frac{1}{2}}} \tag{39}$$

where $\underline{b}$ and $\bar{b}$ are assumed known in advance and $\hat{\omega}^2_{AR}$ is defined in (18). We will refer to Cavaliere and Xu (2011) for details about the algorithm that can be used to simulate the Monte Carlo p-values of the tests. In their paper they also suggest how heteroscedastic shocks can be accounted for. When the bounds $\underline{b}$ and $\bar{b}$ are unknown there are various ways to proceed. For instance the bounds can sometimes be inferred from historical observations or one can conduct a more formal (conservative) testing procedure by taking the minimum of the simulation-based p-values over a grid of admissible bound locations. In a different context, Lundbergh and Teräsvirta (2006) suggest a procedure for estimating implicit bounds, such as inofficial exchange rate target zones inside announced ones, should they exist.

Non-constant volatility. It is generally believed that (mild) heteroscedasticity is a minor issue in unit root testing because the tests allow for heterogenous mixing errors, see e.g. Kim and Schmidt (1993). This applies for the range of tests based on the Phillips-Perron type of unit root tests including the $M-$class of tests, mainly because they are derived in a non-parameteric setting. But actually the parametric counterparts like the augmented (Said) Dickey-Fuller tests *are* robust to some form of heteroscedasticity. Notwithstanding, when volatility is non-stationary the standard unit root results no longer apply. Non-stationary volatility may occur for instance when there is a single or multiple permanent breaks in the volatility process, a property that seems to characterize a wide range of financial time series in particular. Cavaliere (2004) provides a general framework for investigating the effects of permanent changes in volatility on unit root tests.

Some attempts have been made in the literature to alleviate the problems with non-stationary volatility. Boswijk (2001) for instance has proposed a unit root test for the case where volatility is following a nearly integrated GARCH(1,1) process. Kim *et al.* (2002) consider the specific case of of a single abrupt change in variance and suggest a procedure where the breakpoint together with the pre- and post-break variances are first estimated. These are then employed in modified versions of the Phillips-Perron unit root tests. The assumption of a single abrupt change in volatility is, however, not consistent with much empirical evidence which seems to indicate that volatility changes smoothly and that multiple changes in volatility are common when the time series are sufficiently long, see e.g. van Dijk *et al.* (2002) and Amado and Teräsvirta (2012).

Cavaliere and Taylor (2007) propose a methodology that accomodates a fairly general class of volatility change processes. Rather than assuming a specific parametric model for the volatility dynamics they only require that the variance is bounded and implies a countable number of jumps and hence allow both smooth volatility changes and multiple volatility shifts. Based on a consistent estimate of the so-called variance profile they propose a numerical solution to the inference problem by Monte Carlo simulation to obtain the approximate quantiles from the asymptotic distribution of the $M-$class of unit root tests under the null. Their approach can be applied to any of the commonly used unit root tests. We refer to the paper by Cavaliere and Taylor (2007) for details about the numerical procedure. Finally, bootstrap tests for a unit root under non-stationary volatility have been suggested by Chang and Park (2003), Park (2003), Cavaliere and Taylor (2007, 2008, 2009) among others.

9. Empirical illustration

To illustrate different approaches and methodologies discussed in this survey we conduct a small empirical analysis using four macroeconomic time series and a selected number of the tests presented. The time series are the monthly secondary market rate of the 3-month US Treasury bill (T-bill, henceforth) from 1938:1 to 2011:11 (935 observations), the monthly US civilian unemployment rate from 1948:1 to 2011:11 (767 observations), the monthly US CPI inflation from 1947:2 to 2012:1 (780 observations) and the quarterly log transformed US real GDP from 1947:1 to 2011:7 (259 observations). The series are depicted in Figure 1.

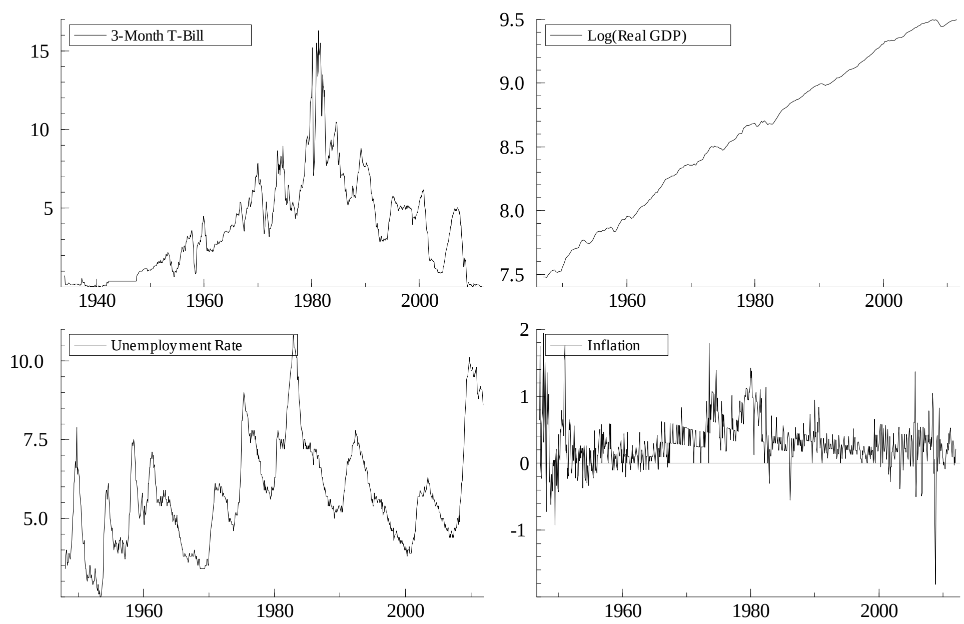

Figure 1. Four macroeconomic time series. The 3-month US T-bill rate (upper left), the log-transformed US real GDP (upper right), the US unemployment rate (lower left) and the US inflation (lower right), respectively. Source: Federal Reserve Economic Data, Federal Reserve Bank of St. Louis, http://research.stlouisfed.org/fred2/.

The individual characteristics of the series look fundamentally different. The log real GDP series has a practically linear deterministic trend, the unemployment rate is a bounded variable, and the inflation rate series contains sharp fluctuations. Estimation of ARMA models (not reported) show that especially the model for the inflation rate series has a strong negative moving average component, thus suggesting that standard ADF and Z tests may suffer from severe size distortions when applied to this series, see e.g. Perron and Ng (1996).

We implement six different tests using both $d_t = 1$ and $d_t = (1,t)'$ as deterministic components. These are the trinity of GLS demeaned/detrended tests ADF_t^{GLS}, $Z_{t_\alpha}^{GLS}$, and $MZ_{t_\alpha}^{GLS}$ defined in section 3 where the lag length k is selected using the MAIC with OLS demeaned/detrended data to avoid power reversal problems, see Perron and Qu (2007). The standard ADF_t, Z_{t_α} and MZ_{t_α} tests are based on the long autoregression in (23) using the same k. While Harvey, Leybourne and Taylor (2009) only consider a union of rejection (UR) strategies for ADF_t^{GLS}, our conjecture is that similar procedures may be defined for the five remaining tests and results from these are reported as well. The results of full sample unit root tests are presented in Table 1.

Full Sample Unit Root Testing

	3-Month T-bill			Log Real GDP		
	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR
ADF	-2.06	-1.77	-1.77	-1.93	-1.45	-1.45
ADF^{GLS}	-1.43	-1.86	-1.86	3.52	-1.05	-1.05
Z_{t_α}	-2.02	-2.01	-2.01	-0.17	-1.71	-1.71
$Z_{t_\alpha}^{GLS}$	-1.73*	-2.10	-1.73*	-0.19	-1.59	-1.59
MZ_{t_α}	-2.01	-2.01	-2.01	-0.17	-1.71	-1.71
$MZ_{t_\alpha}^{GLS}$	-1.73*	-2.09	-1.73*	-0.19	-1.59	-1.59

	Unemployment Rate			Inflation		
	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR
ADF	-2.33	-2.56	-2.33	-11.4***	-11.4***	-11.4***
ADF^{GLS}	-1.06	-2.43	-2.43	-2.61***	-12.5***	-2.61***
Z_{t_α}	-2.55	-2.97	-2.97	-16.31***	-16.24***	-16.31***
$Z_{t_\alpha}^{GLS}$	-1.55	-2.86*	-2.86*	-9.06***	-23.75***	-9.06***
MZ_{t_α}	-2.55	-2.97	-2.97	-13.72***	-13.64***	-13.72***
$MZ_{t_\alpha}^{GLS}$	-1.55	-2.86*	-2.86*	-4.45***	-22.68***	-4.45***

Table 1: Unit root testing of four macroeconomic time series. (*), (**) and (***) denotes significance at a 10%, 5% and 1% level, respectively, based on the critical values from Fuller (1996) for the standard case and from Ng and Perron (2001) for the GLS detrended case.

Several interesting observations can be made from results in Table 1. First, $\alpha = 1$ is clearly rejected for inflation, but the relative (negative) magnitude of the test statistics show the distortions due to a negative moving average component on standard ADF and Z tests. Second, it seems that the correction factor used in the construction of the MZ tests is often negligible, which may well be the result of large sample sizes and strongly consistent estimates of the unit root in the test regressions: often $\hat{\alpha} \approx 0.99$. Finally, we are likely to see an increase in the asymptotic power of the GLS demeaning/detrending procedure applied to the 3-month T-bill and unemployment series since only $Z_{t_\alpha}^{GLS}$ and $MZ_{t_\alpha}^{GLS}$ reject $\alpha = 1$ at the 10% level. Before dismissing a unit root for unemployment it should be noted, however, that the critical values must be altered to reflect the bounds of $\underline{b} = 0$ and $\bar{b} = 100$. Because critical values are becoming numerically larger in this case, a consequence of this is that the unit root hypothesis cannot be rejected at 1% and 5% levels of significance.

9.1 Unit root testing in the recent financial crises

Figure 1 shows that we may have to be concerned with several structural breaks in each series. Nevertheless, for ease of exposition, we shall only treat the sample starting from 1990:1, and we assume that there is a known break in the series caused by the most recent financial crises. We do not analyze inflation, since $\alpha = 1$ is clearly rejected even in the presence of structural breaks. For both the 3-month T-bill and unemployment series, we let T_1 correspond to 2008:9, such that $\lambda_1 \approx 0.85$. Similarly, for the real GDP T_1 corresponds to 2009:1, implying $\lambda_1 \approx 0.89$, since the decline in output seems to occur later corresponding to one quarter.

To emphasize the importance of correcting for a structural break in mean, we consider standard ADF, Z, and MZ tests based on the AO^a and AO^b models. The ADF_t test is based on the auxiliary regression (27), whereas the Z_t and MZ_t tests, following Perron (1990) and Perron and Vogelsang (1992a), are based on the estimate $\hat{\alpha}$ from

$$\tilde{y}_t = \hat{\alpha}\tilde{y}_{t-1} + \hat{d}(1 - L)DU_t + \epsilon_t,$$

where (27) is used to estimate the long run variance $\hat{\omega}_{AR}^2$. We use AIC to select k, since, as argued in section 6.1, it is unclear what the implications of structural breaks are on the MAIC criterion in finite samples. Table 2 shows the three unit root tests both with and without accounting for a structural break.

Subsample Unit Root Testing both with and without Structural Breaks

	3-Month T-bill			Log Real GDP			Unemployment Rate		
No Breaks	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR
ADF	-1.44	-1.79	-1.79	-	-0.453	-0.453	-0.783	-0.955	-0.955
Z_{t_α}	-0.969	-1.88	-1.88	-	-1.43	-1.43	-1.26	-1.47	-1.47
MZ_{t_α}	-0.969	-1.87	-1.87	-	-1.42	-1.42	-1.26	-1.46	-1.46
	3-Month T-bill			Log Real GDP			Unemployment Rate		
Breaks	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR	$d_t = 1$	$d_t = (1,t)'$	UR
ADF	-2.96*	-3.21	-2.96*	-	-3.20	-3.20	-1.40	-3.25	-3.25
Z_{t_α}	-3.05*	-4.13**	-4.13**	-	-3.22	-3.22	-2.24	-3.16	-3.16
MZ_{t_α}	-3.04*	-4.13**	-4.13**	-	-3.15	-3.15	-2.22	-3.16	-3.16

Table 2: Unit root testing of three macroeconomic time series both with and without structural breaks. (*), (**) and (***) denote significance at a 10%, 5% and 1% level, respectively, based on the critical values from Fuller (1996, Appendix 10.A) for the standard case and from Perron (1989, 1990) for the cases with breaks.

It follows from Table 2 that without accounting for a structural break caused by the financial crises, we accept $\alpha = 1$ for all series. However, by allowing for the structural break, we are able to reject $\alpha = 1$ for the 3-month T-bill series, while the values of the test statistics for both the real GDP and unemployment are fairly close to the critical values at a 10% level. While this small analysis is conducted under the simplifying and strong assumption of a single known break, it illustrates the importance of accounting for structural breaks when statistical properties of persistent and trending time series are considered.

10. Conclusion

This paper has provided a selective review of the literature on testing for a unit root. As seen, the field has grown in numerous different directions. New tests have been developed over time to improve power and size properties of previous tests, and testing methodologies and procedures have been extended and modified to allow for an increasing complexity of models with breaks and non-linearities.

Unfortunately, there are many topics that we had to exclude from the exposition even though these areas deserve equal mention. For instance, the use of bootstrap methods for unit root testing in situations where standard assumptions are likely to fail has become a rapidly developing research area during the past decade or so and there still seems to be a potential for new results in this area, see e.g. Cavaliere and Taylor (2007, 2008, 2009), Chang and Park (2003), Paparoditis and Politis (2003), and Park (2003) among others.

Another topic concerns testing when the hypothesis being tested is reversed, i.e. the null of stationarity is tested against the alternatively of a unit root, see e.g. Kwiatkowski, Phillips, Schmidt, and Shin (1992).

In the review of tests for a unit root in the presence of structural breaks it has either been assumed that the break date was known or could be estimated consistently at a sufficiently fast rate. However, this is not always the case and testing for (possibly multiple) breaks in a time series is a separate research area which has attracted a lot of attention. See for instance Perron (2006) for a review and Kejriwal and Perron (2010), and Perron and Yabu (2009a,b) for recent contributions.

Finally, we would like to mention the fractional unit root literature where the order of differencing is fractional rather than being integer valued. Fractionally integrated processes have autocorrelations that die out at a slow hyperbolic rate and hence are often referred to as being long memory. Baillie (1996) provided an early review of this literature but since then the field has grown tremendously. Velasco (2006) provides a recent review of semiparametric estimators of long memory models. More recently there has been an increased focus on how fractional integration and long memory interfere with persistent components caused by structural breaks or non-linearities, see e.g. Diebold and Inoue (2001), Perron and Qu (2010), and Varneskov and Perron (2011). Hence the known problems of discriminating a unit root from structural break models equally apply to fractional long memory models.

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