Tests for the weights of the global minimum variance portfolio in a high-dimensional setting

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Abstract: In this study, we construct two tests for the weights of the global minimum variance portfolio (GMVP) in a high-dimensional setting, namely, when the number of assets p depends on the sample size n such that $\frac{p}{n} \to c \in (0,1)$ as n tends to infinity. The considered tests are based on the sample estimator and on the shrinkage estimator of the GMVP weights. We derive the asymptotic distributions of both test statistics under the null and alternative hypotheses. Moreover, we provide a simulation study where the power functions of the proposed tests are compared with other existing approaches. We observe that the test performs well based on the shrinkage estimator even for values of c close to one.

Keywords: Finance; Portfolio analysis; Global minimum variance portfolio; Statistical test; Shrinkage estimator; Random matrix theory.

1 Introduction

Financial markets have developed rapidly in recent years, and the amount of money invested in risky assets has substantially increased. Due to this, an investor must have knowledge of optimal portfolio proportions in order to receive a large expected return and, at the same time, reduce the level of the risk associated with the investment decision.

Since Markowitz (1952) presented his mean-variance analysis, many works about optimal portfolio selection have been published. However, investors are faced with some difficulties in the practical implementation of these investing theories since sampling error is present when unknown theoretical quantities are estimated.

In classical asymptotic analysis, it is almost always assumed that the sample size increases while the size of the portfolio, namely the number of included assets p, remains constant (e.g.,

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Jobson and Korkie (1981), Okhrin and Schmid (2006)). Nowadays, this case is often called standard asymptotics (see Cam and Yang (2000)). Here, the traditional plug-in estimator of the optimal portfolio, the so-called sample estimator, is consistent and asymptotically normally distributed. However, in many applications, the number of assets in a portfolio is large in comparison to the sample size (i.e. the portfolio dimension p and the sample size n tend to infinity simultaneously) such that $\frac{p}{n}$ tends to the concentration ratio $c > 0$. In this case, we are faced with so-called high-dimensional asymptotics or 'Kolmogorov' asymptotics (see Bühlmann and Van De Geer (2011), Bai and Shi (2011), and Cai and Shen (2011)). Whenever the dimension of the data is large, the classical limit theorems are no longer suitable because the traditional estimators result in a serious departure from the optimal estimators under high-dimensional asymptotics (Bai and Silverstein (2010)). These methods fail to provide consistent estimators of the unknown parameters of the asset returns, that is, the mean vector and the covariance matrix. Generally, the greater the concentration ratio c, the worse the sample estimators are. In these cases, new test statistics must be developed, and completely new asymptotic techniques must be applied for their derivations. Several studies deal with high-dimensional asymptotics in portfolio theory using results from random matrix theory (see Frahm and Jaekel (2008) and Laloux et al. (2000)). Recently, Bodnar, Parolya and Schmid (2017) presented a shrinkage-type estimator for the global minimum variance portfolio (GMVP) weights, and Bodnar, Okhrin and Parolya (2017) derived the optimal shrinkage estimator of the mean-variance portfolio.

Testing the efficiency of a portfolio is a classical problem in finance. The former literature focuses on the case of standard asymptotics or considers exact tests where both p and n are fixed. For example, Gibbons, Ross and Shanken (1989) provided an exact F-test for the efficiency of a given portfolio, and Britten-Jones (1999) derived inference procedures on the efficient portfolio weights based on the application of linear regression. More recently, Bodnar and Schmid (2008) presented a test for the general linear hypothesis of the portfolio weights in the case of elliptically contoured distributions. The contribution of this study is the derivation of statistical techniques for testing the efficiency of a portfolio under high-dimensional asymptotics. Two statistical tests are considered. Whereas the first approach is based on the asymptotic distribution of the test statistic suggested by Bodnar and Schmid (2008) in a high-dimensional setting, the second test makes use of the shrinkage estimator of the GMVP weights and provides a powerful alternative to the existing methods. To the best of our knowledge, this analysis is the first time that the shrinkage approach has been applied to statistical test theory.

It has to be mentioned that there is a direct link between the subject of the paper and classical methods in the statistical signal processing. The equivalent of the GMVP portfolio in signal processing literature is the Capon or minimum variance spatial filter (see, Verdú (1998) and Van Trees (2002)). The estimation risk of the high-dimensional minimum variance beamformer was already studied in Rubio et al. (2012) while its constrained versions it were discussed in Li et al. (2004). The finite sample size effect on minimum variance filter was investigated by Mestre and Lagunas (2006).

The paper is structured as follows. In Section 2, we discuss the main results on distributional properties for optimal portfolio weights presented by Okhrin and Schmid (2006). In Section 3, a new test based on the shrinkage estimator for the GMVP weights is derived, and the high-dimensional version of the test based on the test statistics given in Bodnar and Schmid (2008) is proposed. The asymptotic distributions of the test statistics under both the null hypothesis and the alternative hypothesis are obtained, and the corresponding power functions of both tests are presented. In Section 4, the power functions of the proposed tests are compared with each other for different values of $c \in (0,1)$. In our comparison study, a test of Glombeck (2014) is considered as well. We conclude in Section 5. All proofs are given in the Appendix.

2 Estimation of Optimal Portfolio Weights

We consider a financial market consisting of p risky assets. Let $\mathbf{X}_t$ denote the p-dimensional vector of the returns on risky assets at time t. Suppose that $E(\mathbf{X}_t) = \boldsymbol{\mu}$ and $Cov(\mathbf{X}_t) = \boldsymbol{\Sigma}$. The covariance matrix $\boldsymbol{\Sigma}$ is assumed to be positive definite.

Let us consider a single period investor who invests in the GMVP, one of the most commonly used portfolios (see, for example, Memmel and Kempf (2006), Frahm and Memmel (2010), Okhrin and Schmid (2006), Bodnar and Schmid (2008), Glombeck (2014), and others). This portfolio exhibits the smallest attainable portfolio variance $\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ under the constraint $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1} = (1, \ldots, 1)'$ denotes the p-dimensional vector of ones and $\mathbf{w}$ stands for the vector of portfolio weights. The weights of GMVP are given by

$$\mathbf{w}_{GMVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{1}$$

The global minimum variance portfolio is of fundamental interest in applications involving array signal processing. In the array processing literature it is the so-called minimum variance distortionless response (MVDR) spatial filter or beamformer defined as $\mathbf{w}_{MVDR} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{s}}{\mathbf{s}^H\boldsymbol{\Sigma}^{-1}\mathbf{s}}$ (see, e.g., Van Trees (2002), Chapter 6). The vector $\mathbf{s} \in \mathbb{C}^p$ is the scalar signature vector associated with some waveform $s \in \mathbb{C}$. Thus, the tests for the global minimum variance portfolio developed in this paper could be directly used for minimum variance beamformer just by a simple modification.

The practical implementation of the mean-variance framework in the spirit of Markowitz (1952) relies on estimating the first two moments of the asset returns. Because we do not know the true covariance matrix, it is usually replaced by its sample estimator, which is based on a sample of $n > p$ historical asset returns $\mathbf{X}_1, \ldots, \mathbf{X}_n$ given by

$$\hat{\boldsymbol{\Sigma}}_n = \frac{1}{n-1}\sum_{j=1}^{n}\left(\mathbf{X}_j - \bar{\mathbf{X}}_n\right)\left(\mathbf{X}_j - \bar{\mathbf{X}}_n\right)' \text{ with } \bar{\mathbf{X}}_n = \frac{1}{n}\sum_{v=1}^{n}\mathbf{X}_v. \tag{2}$$

Replacing $\boldsymbol{\Sigma}$ in (1) by the sample estimator $\hat{\boldsymbol{\Sigma}}_n$, we obtain an estimator of the GMVP weights expressed as

$$\hat{\mathbf{w}}_n = \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}. \tag{3}$$

Note that the estimator of the GMVP weights is exclusively a function of the estimator $\hat{\mathbf{\Sigma}}_n$ of the covariance matrix.

Assuming that the asset returns $\{\mathbf{X}_t\}$ follow a stationary Gaussian process with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$, Okhrin and Schmid (2006) proved that the vector of estimated optimal portfolio weights is asymptotically normal. Under the additional assumption of independence, they derived the exact distribution of $\hat{\mathbf{w}}_n$. Okhrin and Schmid (2006) showed that the distribution of arbitrary $p-1$ components of $\hat{\mathbf{w}}_n$ is a $(p-1)$- dimensional t-distribution with $n-p+1$ degrees of freedom and

$$E(\hat{\mathbf{w}}_n) = \mathbf{w}_{GMVP}, Cov(\hat{\mathbf{w}}_n) = \mathbf{\Omega} = \frac{1}{n-p-1}\frac{\mathbf{Q}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}, \mathbf{Q} = \mathbf{\Sigma}^{-1} - \frac{\mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\mathbf{\Sigma}^{-1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}. \tag{4}$$

Consequently, if $\hat{\mathbf{w}}_n^*$ and $\mathbf{w}_{GMVP}^*$ are obtained by deleting the last element of $\hat{\mathbf{w}}_n$ and $\mathbf{w}_{GMVP}$ and if $\mathbf{\Omega}^*$ and $\mathbf{Q}^*$ consist of the first $(p-1)\times(p-1)$ elements of $\mathbf{\Omega}$ and $\mathbf{Q}$, then $\hat{\mathbf{w}}_n^*$ has a $(p-1)$-variate t-distribution with $n-p+1$ degrees of freedom and parameters $\mathbf{w}_{GMVP}^*$ and $\frac{1}{n-p+1}\frac{\mathbf{Q}^*}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$. This distribution is denoted by $\hat{\mathbf{w}}_n^* \sim t_{p-1}(n-p+1, \mathbf{w}_{GMVP}^*, \frac{n-p-1}{n-p+1}\mathbf{\Omega}^*)$, since $\frac{n-p-1}{n-p+1}\mathbf{\Omega}^* = \frac{1}{n-p+1}\frac{\mathbf{Q}^*}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$.

3 Test Theory for the GMVP in High Dimensions

At each time point, an investor is interested to know whether the portfolio he is holding coincides with the true GMVP or has to be reconstructed. For that reason, we consider the following testing problem:

$$H_0 : \mathbf{w}_{GMVP} = \mathbf{r} \qquad \text{against} \qquad H_1 : \mathbf{w}_{GMVP} \neq \mathbf{r}, \tag{5}$$

where $\mathbf{r}$ with $\mathbf{r}'\mathbf{1} = 1$ is a known vector of, for example, the weights of the holding portfolio. Thus, this problem analyses whether the true GMVP weights are equal to some given values.

Bodnar and Schmid (2008) analysed a general linear hypothesis for the GMVP portfolio weights and introduced an exact test assuming that the asset returns are independent and elliptically contoured distributed. Moreover, they derived the exact distribution of the test statistic under the null hypothesis and the alternative hypothesis.

The main focus of this study is high-dimensional portfolios. We want to consider the testing problem (5) in a high-dimensional environment, that is, assuming that $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$. Note that, in this case, H_0 and H_1 depend on n as well. Thus, it would be more precise to write $H_{0,n} : \mathbf{w}_{GMVP,n}^* = \mathbf{r}_n^*$ and $H_{1,n} : \mathbf{w}_{GMVP,n}^* \neq \mathbf{r}_n^*$. In the following, we will ignore this fact in order to simplify our notation. Moreover, it turns out that the sample covariance matrix is no longer a good estimator of the covariance matrix (see Bai and

Silverstein (2010) and Bai and Shi (2011)). For that reason, it is unclear how well the test of Bodnar and Schmid (2008) behaves in that context. First, we study its behaviour under the high-dimensional asymptotics, and, after that, we propose an alternative test that makes use of the shrinkage estimator for the portfolio weights (cf. Bodnar, Parolya and Schmid (2017)).

In recent years, several studies have dealt with estimators of unknown portfolio parameters under high-dimensional asymptotics with applications to portfolio theory. Glombeck (2014) formulated tests for the portfolio weights, variances of the excess returns, and Sharpe ratios of the GMVP for $c \in (0,1)$. Bodnar, Parolya and Schmid (2017) and Bodnar, Okhrin and Parolya (2017) derived the shrinkage estimators for the GMVP and for the mean-variance portfolio, respectively, under the Kolmogorov asymptotics for $c \in (0,\infty)$.

3.1 A Test Based on the Mahalanobis Distance

Bodnar and Schmid (2008) proposed a test for a general linear hypothesis of the weights of the global minimum variance portfolio. Here, we are interested in the special case (5). For this case, the test statistic is given by

$$T_n = \frac{n-p}{p-1}(\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1})(\hat{\mathbf{w}}_n^* - \mathbf{r}^*)'(\hat{\mathbf{Q}}_n^*)^{-1}(\hat{\mathbf{w}}_n^* - \mathbf{r}^*), \tag{6}$$

where $\hat{\mathbf{Q}}_n^*$ consists of the first $(p-1)\times(p-1)$ elements of $\hat{\mathbf{Q}}_n = \hat{\mathbf{\Sigma}}_n^{-1} - \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}$ and the number of assets p in the portfolio is fixed. It was shown that, under the null hypothesis, $T_n \sim F_{p-1,n-p}$. Moreover, the density of T_n under the alternative hypothesis H_1 is equal to

$$\begin{aligned} f_{T_n}(x) &= f_{p-1,n-p}(x)\,(1+\lambda)^{-(n-1)/2} \\ &\times \quad {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{p-1}{2}; \frac{(p-1)x}{n-p+(p-1)x}\frac{\lambda}{1+\lambda}\right), \end{aligned} \tag{7}$$

where

$$\lambda = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}(\mathbf{w}_{GMVP}^* - \mathbf{r}^*)'(\mathbf{Q}^*)^{-1}(\mathbf{w}_{GMVP}^* - \mathbf{r}^*) \tag{8}$$

and ${}_2F_1$ stands for the hypergeometric function (see Abramowitz and Stegun (1964), chap. 15), that is,

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{i=0}^{\infty}\frac{\Gamma(a+i)\Gamma(b+i)}{\Gamma(c+i)}\frac{z^i}{i!}.$$

Thus, the exact power function of the test is known. Note that this result is also valid for matrix-variate elliptically contoured distributions (see Bodnar and Schmid (2008)). On the other hand, several computational difficulties appear when the power function of the test is calculated for large values of p and n, since doing so involves a hypergeometric function whose computation is very challenging for large values of p and n. In order to deal with this problem, we derive the asymptotic distribution of T_n in a high-dimensional setting. This result is given in Theorem 1. The proof is in the Appendix. Since λ depends on p (i.e. on n) through $\mathbf{\Sigma}$, we write λ_n in the rest of the paper.

Theorem 1 *Let $p \equiv p(n)$ and $c_n = \frac{p}{n} \to c \in (0,1)$. Assume that $\{\mathbf{X}_t\}$ is a sequence of independent and normally distributed p-dimensional random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$, which is assumed to be positive definite. Let*

$$C_n^2 = 2 + 2\frac{\lambda_n^2}{c} + 4\frac{\lambda_n}{c} + 2\frac{c}{1-c}\left(1 + \frac{\lambda_n}{c}\right)^2 .$$

Then, it holds that

$$\sqrt{p-1}\left(\frac{T_n - 1 - \lambda_n \frac{n-1}{p-1}}{C_n}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

for $p/n \to c \in (0,1)$ as $n \to \infty$. Under the null hypothesis, $\sqrt{p-1}\,(T_n - 1) \xrightarrow{d} \mathcal{N}(0, 2/(1-c))$ for $p/n \to c \in (0,1)$ as $n \to \infty$.

The results of Theorem 1 lead to an asymptotic expression of the power function given by

$$\begin{aligned} & P\left(\frac{\sqrt{p-1}\,(T_n - 1)}{\sqrt{2/(1-c)}} > z_{1-\alpha}\right) \\ = \;& 1 - P\left(\frac{\sqrt{p-1}\left(T_n - 1 - \lambda_n \frac{n-1}{p-1}\right)}{C_n} \le \frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{p-1}\lambda_n \frac{n-1}{p-1}}{C_n}\right) \\ \approx \;& 1 - \Phi\left(\frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{p-1}\frac{\lambda_n}{c}}{C_n}\right), \end{aligned} \tag{9}$$

where $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

In Figure 1, we plot the power function (9) as a function of λ_n for several values of c and n (solid line). In addition, the empirical power of the test is shown for the same values of c and n (dashed line) and is equal to the relative number of rejections of the null hypothesis obtained via a simulation study. It is remarkable that, following the proof of Theorem 1, the considered simulation study can be considerably simplified. Instead of generating a $p \times n$ random matrix of asset returns in each simulation run, we simulate four independent random variables from standard univariate distributions and then compute the statistic T_n for the given value of λ_n following the stochastic representation (25) in the Appendix. Namely, the simulation study is performed in the following way:

(i) Generate four independent random variables $\omega_1^{(b)} \sim \mathcal{N}(0,1)$, $\xi_2^{(b)} \sim \chi^2_{n-p}$, $\xi_3^{(b)} \sim \chi^2_{n-1}$, and $\xi_4^{(b)} \sim \chi^2_{p-2}$

(ii) For fixed λ_n, compute

$$T_n^{(b)} \stackrel{d}{=} \frac{n-p}{p-1}\frac{(\sqrt{\lambda_n \xi_3^{(b)}} + \omega_1^{(b)})^2 + \xi_4^{(b)}}{\xi_2^{(b)}}$$

(iii) Repeat steps (a) and (b) for $b = 1, ..., B$ and calculate the empirical power by

$$\hat{P} = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}_{(z_{1-\alpha}, +\infty)} \left(\frac{\sqrt{p-1}\left(T_n^{(b)} - 1\right)}{\sqrt{2/(1-c)}} \right), \tag{10}$$

where $\mathbb{1}_{\mathcal{A}}(.)$ is the indicator function of the set $\mathcal{A}$.

In Figure 1, we observe a good performance of the asymptotic approximation of the power function. This approximation works perfectly for both small and large values of c.

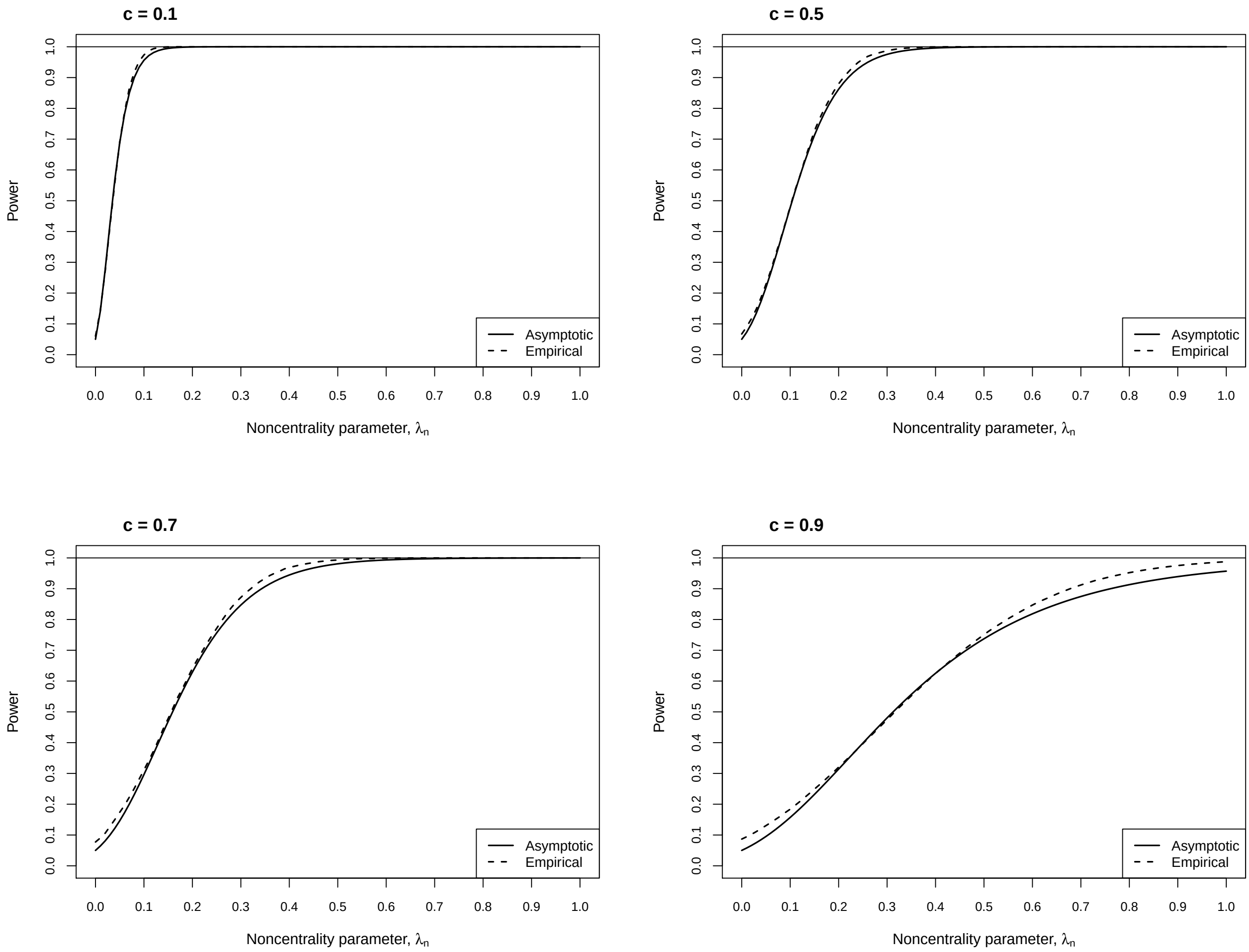


Figure 1: Asymptotic power function vs. empirical power function for different values as a function of λ_n in (7) for various values of $c \in \{0.1, 0.5, 0.7, 0.9\}$ and a 5% significance level.

3.2 Test Based on a Shrinkage Estimator

In most cases, the unknown parameters of the asset return distribution are replaced by their sample counterparts when an optimal portfolio is constructed. In recent years, however, other types of estimators, such as shrinkage estimators, have been discussed as well (see Okhrin and Schmid (2007) and Bodnar, Parolya and Schmid (2017)). The shrinkage methodology was introduced by Stein (1956). His results were extended by Efron and Morris (1976) to the case in which the covariance matrix is unknown. The shrinkage methodology can be applied to the expected asset returns (e.g. Jorion (1986)) and the covariance matrix (Bodnar, Gupta and Parolya (2014, 2016)). Both of these applications appear to be very successful in reducing damaging influences on the portfolio selection. A shrinkage estimator was applied directly to the portfolio weights by Golosnoy and Okhrin (2007) and Okhrin and Schmid (2008). They showed that the shrinkage estimators of the portfolio weights lead to a decrease in the variance of the portfolio weights and to an increase in utility.

Bodnar, Parolya and Schmid (2017) proposed a new shrinkage estimator for the weights of the GMVP that turns out to provide better results in the high-dimensional case than the existing estimators do. This estimator is based on a convex combination of the sample estimator of the GMVP weights and an arbitrary constant vector expressed as

$$\hat{\mathbf{w}}_{n;GSE} = \alpha_n \frac{\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}} + (1-\alpha_n)\mathbf{b}_n \qquad \text{with} \qquad \mathbf{b}_n'\mathbf{1} = 1. \tag{11}$$

Here, the index GSE stands for 'general shrinkage estimator'. It is assumed that $\mathbf{b}_n \in \mathbb{R}^p$ is a vector of constants such that $\mathbf{b}_n'\mathbf{\Sigma}\mathbf{b}_n$ is uniformly bounded. Bodnar, Parolya and Schmid (2017) proposed determining the optimal shrinkage intensity α_n for a given target portfolio $\mathbf{b}_n$ such that the out-of-sample risk is minimal, that is,

$$L = (\hat{\mathbf{w}}_{n;GSE} - \mathbf{w}_{GMVP})'\mathbf{\Sigma}(\hat{\mathbf{w}}_{n;GSE} - \mathbf{w}_{GMVP}) \tag{12}$$

is minimized with respect to α_n. This result leads to

$$\hat{\alpha}_n = \frac{(\mathbf{b}_n - \hat{\mathbf{w}}_n)'\,\mathbf{\Sigma}\,\mathbf{b}_n}{(\mathbf{b}_n - \hat{\mathbf{w}}_n)'\,\mathbf{\Sigma}\,(\mathbf{b}_n - \hat{\mathbf{w}}_n)}. \tag{13}$$

The authors showed that the optimal shrinkage intensity $\hat{\alpha}_n$ is almost surely asymptotically equivalent to a non-random quantity $\tilde{\alpha}_n \in [0,1]$ when $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$, which is given by

$$\tilde{\alpha}_n = \frac{(1-c)R_{\mathbf{b}_n}}{c + (1-c)R_{\mathbf{b}_n}}, \tag{14}$$

where

$$R_{\mathbf{b}_n} = \frac{\sigma_{\mathbf{b}_n}^2 - \sigma_n^2}{\sigma_n^2} = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{b}_n'\mathbf{\Sigma}\mathbf{b}_n - 1$$

is the relative loss of the target portfolio $\mathbf{b}_n$, $\sigma_{\mathbf{b}_n}^2 = \mathbf{b}_n'\mathbf{\Sigma}\mathbf{b}_n$ is the variance of the target portfolio, and $\sigma_n^2 = 1/\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$ is the variance of the GMVP. This result provides an estimator of the optimal shrinkage intensity given by

$$\hat{\hat{\alpha}}_n = \frac{(1-\frac{p}{n})\hat{R}_{\mathbf{b}_n}}{\frac{p}{n} + (1-\frac{p}{n})\hat{R}_{\mathbf{b}_n}}, \;\; \hat{R}_{\mathbf{b}_n} = (1-\frac{p}{n})\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1} - 1. \tag{15}$$

Using the estimated shrinkage intensity $\hat{\tilde{\alpha}}_n$, the corresponding portfolio weights are given by

$$\hat{\mathbf{w}}_{n;ESI} = \hat{\tilde{\alpha}}_n \hat{\mathbf{w}}_n + (1 - \hat{\tilde{\alpha}}_n)\mathbf{b}_n. \tag{16}$$

Bodnar, Parolya and Schmid (2017) proved that the ratio $\dfrac{\hat{\tilde{\alpha}}_n}{\tilde{\alpha}_n} \to 1$ if $\frac{p}{n} \to c \in (0,1)$ as $n \to \infty$. In Theorem 2, we show that the estimated intensity is asymptotically normally distributed. The proof of Theorem 2 is given in the Appendix.

Theorem 2 *Let $p \equiv p(n)$ and $c_n = \frac{p}{n} \to c \in (0,1)$. Assume that $\{\mathbf{X}_t\}$ is a sequence of independent and normally distributed p-dimensional random vectors with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, which is assumed to be positive definite. Then*

$$\sqrt{n}\frac{\hat{\tilde{\alpha}}_n - A_n}{B_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{for } p/n \to c \in (0,1) \text{ as } n \to \infty, \tag{17}$$

where

$$\begin{aligned} A_n &= \frac{(1-c_n)R_{\mathbf{b}_n}}{c_n + (1-c_n)R_{\mathbf{b}_n}}, \\ B_n^2 &= 2\frac{c_n^3(1-c_n)(1+R_{\mathbf{b}_n})^2}{(c_n + (1-c_n)R_{\mathbf{b}_n})^4}. \end{aligned}$$

Next, we introduce a test based on the estimated shrinkage intensity. The motivation is based on the following equivalences:

$$\tilde{\alpha}_n = 0 \iff R_{\mathbf{b}_n} = 0 \iff \sigma^2_{\mathbf{b}_n} = \sigma^2_n.$$

This result means that $\tilde{\alpha}_n = 0$ if and only if the variance of the portfolio based on $\mathbf{b}_n$ is equal to the variance of the GMVP. This finding in turn means that $\mathbf{b}_n'\boldsymbol{\Sigma}\mathbf{b}_n = 1/\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} = \min_{\mathbf{w}:\, \mathbf{w}'\mathbf{1}=1} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \mathbf{w}_{GMVP}'\boldsymbol{\Sigma}\,\mathbf{w}_{GMVP}$. Since the GMVP weights are uniquely determined, this result is valid if and only if $\mathbf{b}_n = \mathbf{w}_{GMVP}$. Choosing $\mathbf{b}_n = \mathbf{r}$, it holds that

$$\mathbf{w}_{GMVP} = \mathbf{r} \iff \tilde{\alpha}_n = 0.$$

Thus, it is possible to obtain a test for the structure of the GMVP using the shrinkage intensity with the hypothesis given by

$$H_0 : \tilde{\alpha}_n = 0 \qquad \text{against} \qquad H_1 : \tilde{\alpha}_n > 0. \tag{18}$$

Note that $\hat{\tilde{\alpha}} = \hat{\tilde{\alpha}}(\mathbf{b}_n)$. Let $S_n = \sqrt{n}\,\hat{\tilde{\alpha}}(\mathbf{b}_n = \mathbf{r})$. For testing (18), we use the test statistic S_n.

Theorem 3 *Suppose that the conditions of Theorem 2 are satisfied. Then*

$$\frac{S_n - \sqrt{n}A_n}{B_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{for } p/n \to c \in (0,1) \text{ as } n \to \infty,$$

where A_n *and* B_n *are given in the statement of Theorem 2. Under the null hypothesis,* $S_n \xrightarrow{d} \mathcal{N}(0, 2(1-c)/c)$ *for* $p/n \to c \in (0,1)$ *as* $n \to \infty$ *.*

The proof of Theorem 3 follows directly from Theorem 2. This result gives us a promising new approach for detecting deviations of the true portfolio weights from the given quantities. Using Theorem 3, we are able to make a statement about the power function of this test. Since A_n and B_n depend on $\mathbf{b}_n$, we only have to replace this quantity with $\mathbf{r}$. It holds that

$$P\left(\frac{S_n}{\sqrt{2\frac{1-c}{c}}} > z_{1-\alpha}\right) = 1 - P\left(\frac{S_n - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})} \leq \frac{\sqrt{2\frac{1-c}{c}}z_{1-\alpha} - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})}\right)$$
$$\approx 1 - \Phi\left(\frac{\sqrt{2\frac{1-c}{c}}z_{1-\alpha} - A_n(\mathbf{b}_n = \mathbf{r})}{B_n(\mathbf{b}_n = \mathbf{r})}\right). \quad (19)$$

Note that the approximation given in (19) is purely a function of $R_{\mathbf{b}_n = \mathbf{r}}$. This property is a main difference from the test discussed in Section 3.1, where the power function is a function of λ_n. These properties are very useful to analyse the performances of both tests and simplify the power analysis.

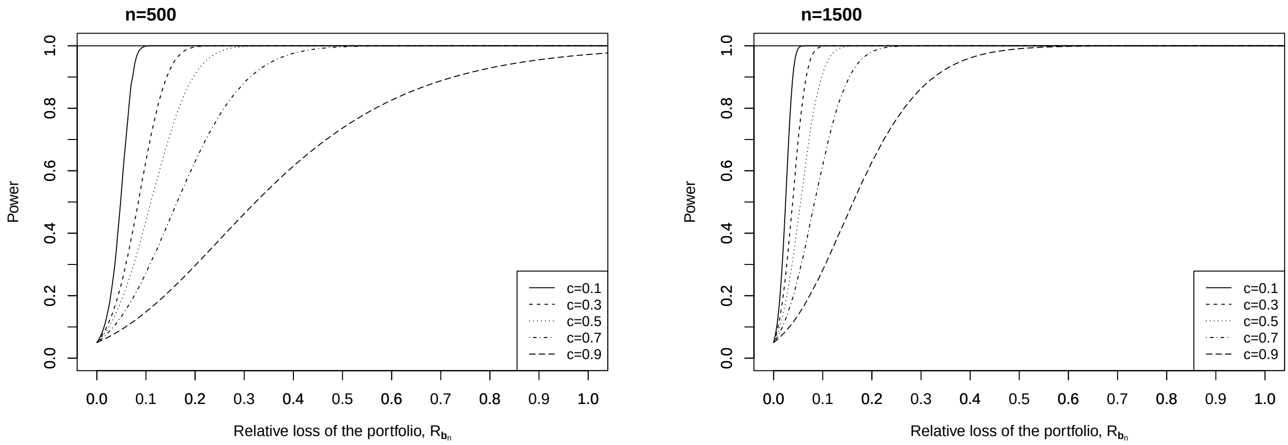


Figure 2: The asymptotic power function of the test in (18) as a function of $R_{\mathbf{b}_n}$ for various values of $c \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and a 5% significance level. The number of observations is $n = 500$.

In *Figure 3*, the power of the test is shown as a function of $R_{\mathbf{b}_n}$ and n. It can be seen that the test performs better for smaller values of c. With increasing values of c, the power of the test decreases. We determine the power function for two different sample numbers, $n = 500$ and $n = 1500$. As expected, the test shows a better performance for larger values of n, since $A_n(\mathbf{b}_n = \mathbf{r})$ increases, the numerator of the expression in the cumulative distribution function in (19) becomes negative, and the whole expression tends to one.

4 Comparison Study

The aim of this section is to compare several tests for the weights of the GMVP.

In the preceding two subsections, we considered two tests for the weights of the GMVP. For the test based on the empirical portfolio weights, the exact distribution of the test statistic is known. In Section 3.1, the asymptotic power function of the test proposed by Bodnar and Schmid (2008) is derived in a high-dimensional setting. In Section 3.2, a new test is proposed, and its asymptotic power function, which purely depends on $R_{\mathbf{b}_n=\mathbf{r}}$, is determined. The fact that both tests depend on different quantities complicates the comparison of both tests. Note that

$$R_{\mathbf{b}_n=\mathbf{r}} = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\ \mathbf{r}'\mathbf{\Sigma}\mathbf{r} - 1 = \lambda_n \frac{\mathbf{r}'\mathbf{\Sigma}\mathbf{r}}{(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\mathbf{Q}^*)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)} - 1.$$

In Section 4.1, both tests are compared with each other. Additionally, we include the test presented by Glombeck (2014, Theorem 10) in our comparison study.

4.1 Design of the Comparison Study

Let $\mathbf{\Sigma}$ be a $p \times p$ positive definite covariance matrix of asset returns, n the number of samples, and $p \equiv p(n)$. The structure of the covariance matrix is chosen in the following way: one-ninth of the eigenvalues are set equal to 2, four-ninths are set equal to 5, and the rest are set equal to 10. In doing so, we can ensure that the eigenvalues are not very dispersed, and if p increases, then the spectrum of the covariance matrix does not change its behaviour. Then, the covariance matrix is determined as follows

$$\mathbf{\Sigma} = \Theta \mathbf{\Lambda} \Theta',$$

where $\mathbf{\Lambda}$ is the diagonal matrix whose diagonal elements are the predefined eigenvalues and Θ is the $p \times p$ matrix of eigenvectors obtained from the spectral decomposition of a standard Wishart-distributed random matrix.

We consider the following scenario for modelling the changes. Under the alternative hypothesis, the covariance matrix is defined by

$$\mathbf{\Sigma}_1 = \Delta \mathbf{\Sigma} \Delta, \tag{20}$$

where Δ denotes the change in the standard deviation and is given by

$$\Delta = \left(\begin{array}{c|c} D_m & \mathbf{0} \\ \hline \mathbf{0} & I_{p-m} \end{array} \right), \tag{21}$$

with $D_m = diag(a)$ and $a = 1 + 0.1k$, $k \in \{1, 2, \dots, 15\}$, and $m \in \{0.1p, 0.2p, 0.5p, 0.8p\}$.

In order to demonstrate the influence of Δ, we build the ℓ_1 norm of the difference between $\mathbf{w}_{GMVP}(\mathbf{\Sigma}_1)$ and $\mathbf{w}_{GMVP}(\mathbf{\Sigma})$ as a function of a. This difference relates to the proportional transaction costs for moving to the new optimal portfolio weights $\mathbf{w}_{GMVP}(\mathbf{\Sigma}_1)$. In *Figure 3*, we can see that the largest influence on the portfolio composition is observed if $m = 0.5p$. For m larger than $0.5p$, this influence decreases. The results obtained for this scenario present an almost linear relationship between the ℓ_1 norm of the difference vector and the size of the change. It is worth mentioning that the differences are all zero when no changes occur (i.e. under H_0).

4.2 Comparison of the Tests

In this section, we present the results of a simulation study to compare the powers of the three tests. The simulation is based on 10^5 independent realizations of Δ . The significance level α is chosen to be 5% , and the concentration ratio c takes a value within the set $\{0.1, 0.5, 0.7, 0.9\}$.

In order to illustrate the performance of the tests based on the shrinkage approach, the test based on the statistic of Bodnar and Schmid (2008), and the test proposed by Glombeck (2014), the empirical power functions for the general hypothesis are evaluated for $m = 0.2p$ (*Figure 4a*) and $m = 0.5p$ (*Figure 4b*).

In *Figure 4a*, where 20% of the elements in the main diagonal of the covariance matrix are contaminated, we observe a slow increase of the power functions for $c = 0.9$ and better behaviour for smaller values of c . In the case $c = 0.1$, there is no significant difference in the performance of the tests. For $c = 0.5$ and $c = 0.7$, the power curves of Glombeck's test and the test of Bodnar and Schmid (2008) are very close to each other, whereas the test presented by Glombeck (2014) shows a slightly better performance than the one presented by Bodnar and Schmid (2008) when $c = 0.9$. For $c = \{0.5, 0.7, 0.9\}$, the test based on the shrinkage approach outperforms its competitors.

Figure 4b illustrates the behaviour of the tests for $m = 0.5p$. We detect an improvement in the performances of the tests for all values of c . In the case $c = 0.1$, the test of Bodnar and Schmid (2008) outperforms both the shrinkage approach and Glombeck's test, whereas, for $c = 0.9$, this test appears to be the worst one. For $c = 0.5$ and $c = 0.7$, the same situation occurs as when $m = 0.2p$, where the powers of Glombeck's test and the test of Bodnar and Schmid (2008) almost coincide with each other. The test based on the estimated shrinkage intensity lies above the rest of the competitors for $c = 0.5$, $c = 0.7$, and $c = 0.9$.

5 Summary

The main focus of this study is the inference of the GMVP weights. After constructing an optimal portfolio, an investor is interested to know whether or not the weights of the portfolio he is holding are still optimal at a fixed time point. For that reason, we investigate several asymptotic and exact statistical procedures for detecting deviations in the weights of the GMVP. One test is based on the sample estimator of the GMVP weights, whereas another uses its shrinkage estimator. To the best of our knowledge, the shrinkage approach, which is very popular in point estimation, is applied in test theory for the first time. The asymptotic distributions of both test statistics are obtained under the null and alternative hypotheses in a high-dimensional setting. This finding is a great advantage with respect to other approaches that appear in the literature which do not elaborate on the distribution under the alternative hypothesis (e.g. Glombeck (2014)).

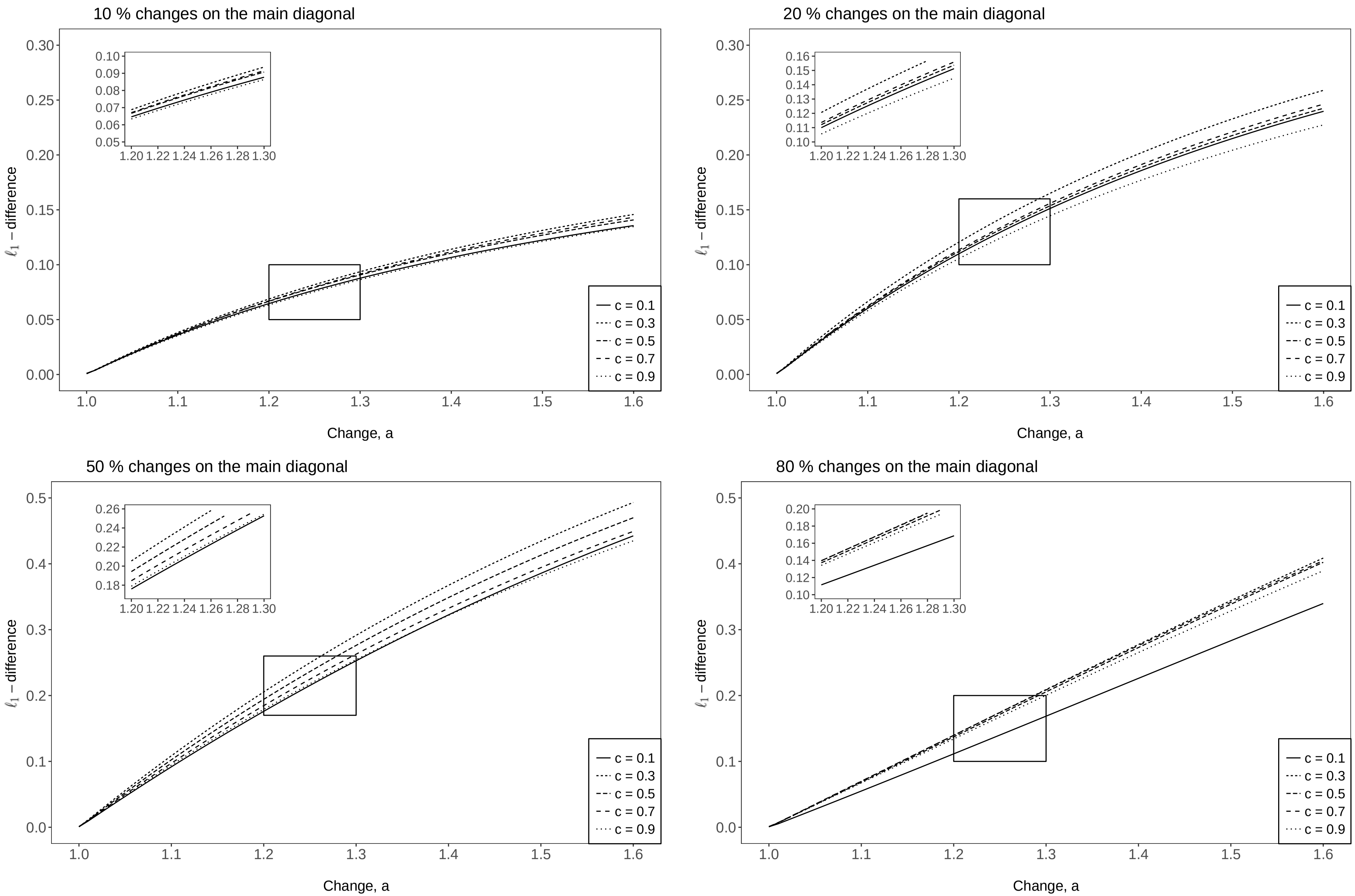


Figure 3: ℓ_1 - differences $\sum_{i=1}^{p} \left| w_{i,GMVP}(\mathbf{\Sigma}_1) - w_{i,GMVP}(\mathbf{\Sigma}) \right|$ as a function of change, a . Here, $n = 500$.

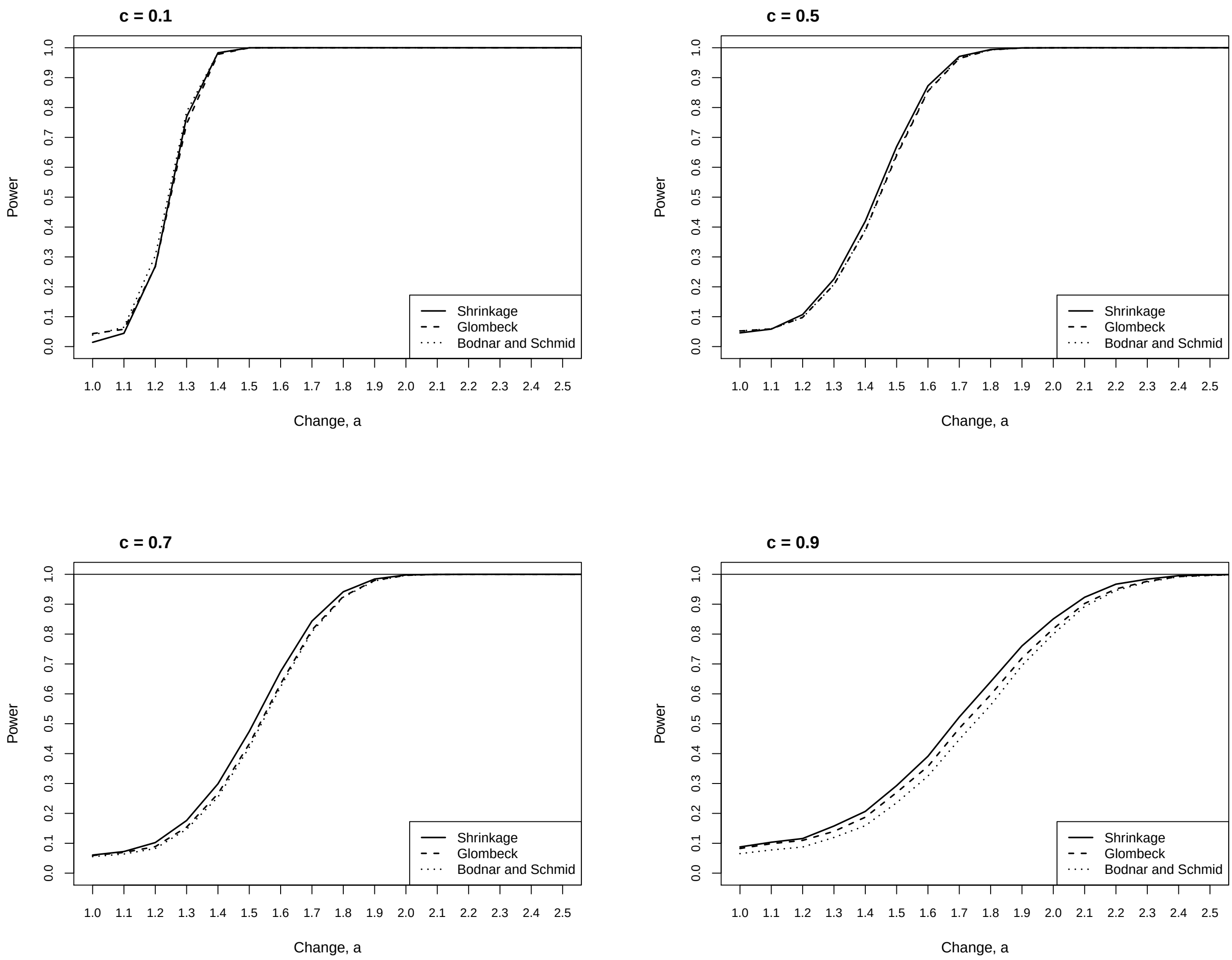
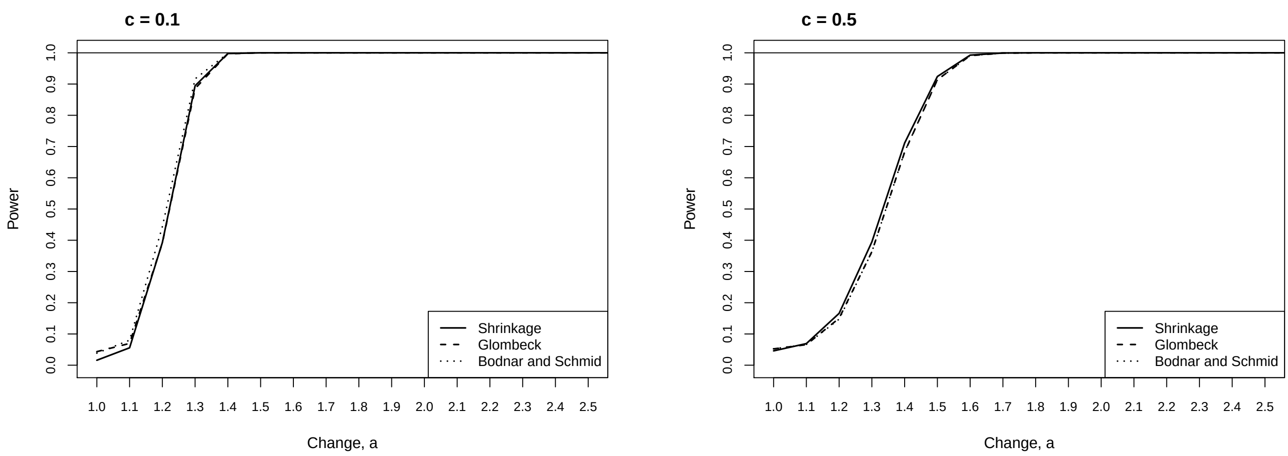


Figure 4a: Empirical power functions of the three tests for different values of c and 20% changes on the main diagonal according to scenario given in (20) and $n = 500$.



In order to compare the performances of the proposed procedures, the empirical power functions of the derived tests are determined. It is shown that the test based on the shrinkage approach performs uniformly better than the other tests considered in the analysis for moderate

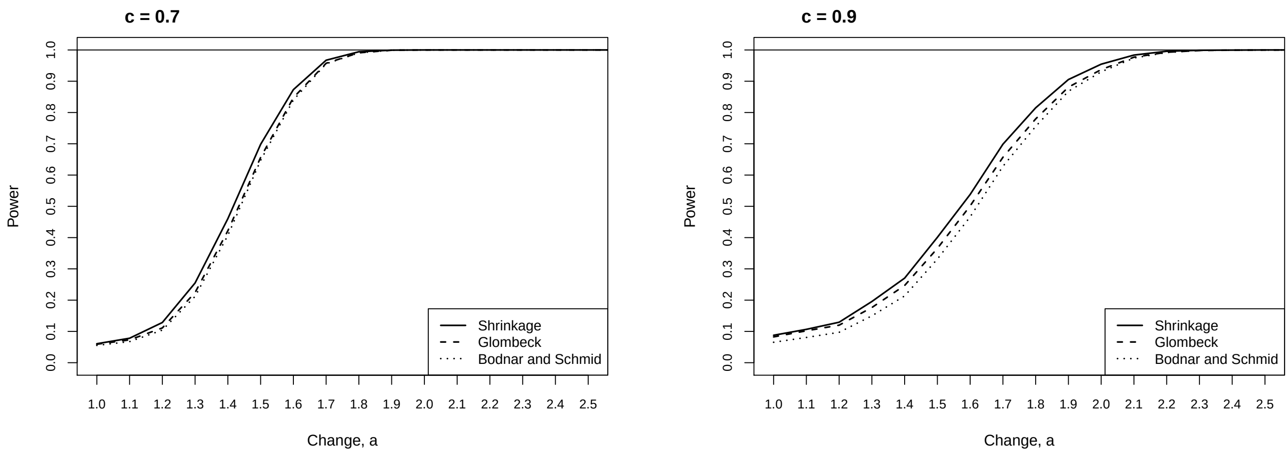


Figure 4b: Empirical power functions of the three tests for different values of c and 50% changes on the main diagonal according to the scenario given in (20) and $n = 500$.

and large values of the concentration ratio c. This approach seems to be very promising for testing portfolio weights in a high-dimensional situation.

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Appendix

In this section, the proof of Theorems 1 and 2 are given.

Proof of Theorem 1:

We first derive a stochastic representation for T_n. Let the symbol $\stackrel{d}{=}$ denote equality in distribution. Then, it holds that (see, the proof of Theorem 2 in Bodnar and Schmid (2008))

$$T_n \stackrel{d}{=} \frac{n-p}{p-1}\frac{\xi_1}{\xi_2}, \tag{22}$$

where

$$\xi_2 = (n-1)\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} \sim \chi^2_{n-p}, \tag{23}$$

$$\xi_1|\hat{\mathbf{Q}}^*_n = (n-1)\left(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\right)(\hat{\mathbf{w}}^*_n - \mathbf{r}^*)'(\hat{\mathbf{Q}}^*_n)^{-1}(\hat{\mathbf{w}}^*_n - \mathbf{r}^*) \sim \chi^2_{p-1,\lambda_n(\hat{\mathbf{Q}}^*_n)},$$

with

$$\lambda_n(\hat{\mathbf{Q}}^*_n) = (n-1)\left(\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\right)(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\hat{\mathbf{Q}}^*_n)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*),$$

and ξ_2 is independent of both $\hat{\mathbf{Q}}^*_n$ and $(\hat{\mathbf{Q}}^*_n)^{-1}(\hat{\mathbf{w}}^*_n - \mathbf{r}^*)$. Moreover, in using $(n-1)(\hat{\mathbf{Q}}^*_n)^{-1} \sim \mathcal{W}_p(n-1,(\mathbf{Q}^*)^{-1})$ (cf. Muirhead (1982, Theorems 3.2.10 and 3.2.11)), we obtain

$$\lambda_n(\hat{\mathbf{Q}}^*_n) = \lambda_n \frac{(n-1)(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\hat{\mathbf{Q}}^*_n)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)}{(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)'(\mathbf{Q}^*)^{-1}(\mathbf{w}^*_{GMVP} - \mathbf{r}^*)} \sim \lambda_n\xi_3, \tag{24}$$

where $\xi_3 \sim \chi^2_{n-1}$.

The last equality shows that the conditional distribution of ξ_1 given $\hat{\mathbf{Q}}^*_n$ depends only on $\hat{\mathbf{Q}}^*_n$ over ξ_3, and, consequently, the conditional distribution $\xi_1|\hat{\mathbf{Q}}^*_n$ coincides with $\xi_1|\xi_3$. Using the distributional properties of the non-central F-distribution, we obtain the following stochastic representation for ξ_1 given by

$$\xi_1 \overset{d}{=} (\sqrt{\lambda_n\xi_3} + \omega_1)^2 + \xi_4,$$

and, hence, the stochastic representation of T_n is expressed as

$$T_n \overset{d}{=} \frac{n-p}{p-1}\frac{(\sqrt{\lambda_n\xi_3} + \omega_1)^2 + \xi_4}{\xi_2}, \tag{25}$$

where $\omega_1 \sim \mathcal{N}(0,1)$, $\xi_2 \sim \chi^2_{n-p}$, $\xi_3 \sim \chi^2_{n-1}$, and $\xi_4 \sim \chi^2_{p-2}$; ω_1, ξ_2, ξ_3, and ξ_4 are independent.

Applying (25), we obtain

$$\begin{aligned}
&\frac{n-p}{\xi_2}\sqrt{p-1}\left(\frac{\lambda_n\xi_3 + 2\sqrt{\lambda_n\xi_3}\omega_1 + \omega_1^2 + \xi_4}{p-1} - \left(1 + \lambda_n\frac{n-1}{p-1}\right)\frac{\xi_2}{n-p}\right)\\
= &\frac{n-p}{\xi_2}\left(\lambda_n\frac{n-1}{p-1}\sqrt{p-1}\left(\frac{\xi_3}{n-1} - 1\right) + \sqrt{p-1}\left(\frac{\xi_4}{p-1} - 1\right)\right.\\
- &\left.\left(1 + \lambda_n\frac{n-1}{p-1}\right)\sqrt{p-1}\left(\frac{\xi_2}{n-p} - 1\right) + 2\sqrt{\lambda_n}\sqrt{\frac{\xi_3}{p-1}}\omega_1 + \frac{\omega_1^2}{\sqrt{p-1}}\right).
\end{aligned}$$

Using the asymptotic properties of a χ^2-distribution with infinite degrees of freedom and the independence of ω_1, ξ_2, ξ_3, ξ_4, the application of Slutsky's lemma (see, for example, Theorem 1.5 in DasGupta (2008)) leads to

$$\sqrt{p-1}\left(\frac{T_n - 1 - \lambda_n \frac{n-1}{p-1}}{C_n}\right) \xrightarrow{d} \mathcal{N}\left(0,1\right),$$

where

$$C_n^2 = 2 + 2\frac{\lambda_n^2}{c} + 4\frac{\lambda_n}{c} + 2\frac{c}{1-c}\left(1+\frac{\lambda_n}{c}\right)^2.$$

□

Proof of Theorem 2:

In order to stress the dependence on n, we use the notation $\mathbf{\Sigma}_n$ in the following.

a) Using Proposition 3 of Glombeck (2014), we have

$$\sqrt{n}\begin{pmatrix} \frac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n} - 1 \\ \frac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}} - \frac{1}{1-c_n} \end{pmatrix} \xrightarrow{d} \mathcal{N}\left[\begin{pmatrix}0\\0\end{pmatrix}, 2\begin{pmatrix} 1 & -\frac{1}{1-c} \\ -\frac{1}{1-c} & \frac{1}{(1-c)^3}\end{pmatrix}\right], \tag{26}$$

if $n\to\infty$. We can rewrite $\hat{R}_{\mathbf{b}_n}$ as

$$\begin{aligned}
\hat{R}_{\mathbf{b}_n} &= (1-c_n)\frac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n}\frac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1} - 1 \\
&= \Delta_n(1-c_n)(D_nE_n + \frac{D_n}{1-c_n} + E_n) + \Delta_n - 1
\end{aligned}$$

with

$$\Delta_n = \mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n\ \mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}, \quad D_n = \frac{\mathbf{b}_n'\hat{\mathbf{\Sigma}}_n\mathbf{b}_n}{\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n} - 1, \quad E_n = \frac{\mathbf{1}'\hat{\mathbf{\Sigma}}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}} - \frac{1}{1-c_n}.$$

Using (26), it follows that

$$\begin{aligned}
\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n} - \Delta_n + 1}{\Delta_n} &= \sqrt{n}(1-c_n)(D_nE_n + D_n/(1-c_n) + E_n) \\
&= (1-c_n)\sqrt{n}(D_n/(1-c_n) + E_n) + o_p(1) \\
&= \begin{pmatrix}1 & 1-c_n\end{pmatrix}\sqrt{n}\begin{pmatrix}D_n\\E_n\end{pmatrix} + o_p(1) \xrightarrow{d} \mathcal{N}\left(0, 2\frac{c}{1-c}\right)
\end{aligned}$$

since $\sqrt{n}D_n \xrightarrow{d} \mathcal{N}(0,2)$ and $\sqrt{n}E_n \xrightarrow{d} \mathcal{N}(0, \frac{2}{(1-c)^3})$.

b) Since

$$\hat{\hat{\alpha}}_n = \frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n} + \frac{\Delta_n - 1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n} + (1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n} + \frac{\Delta_n-1}{\Delta_n}\right)},$$

it follows that

$$\sqrt{n}\left(\hat{\hat{\alpha}}_n - \frac{(1-c_n)(\Delta_n-1)}{c_n+(1-c_n)(\Delta_n-1)}\right) = I_n + II_n$$

with

$$\begin{aligned}
I_n &= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}\\
&= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}\quad\frac{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}\\
&= \sqrt{n}\,\frac{(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\right)}{1-c_n-\frac{1-2c_n}{\Delta_n}}\quad\frac{1}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
II_n &= \sqrt{n}(1-c_n)(1-\frac{1}{\Delta_n})\left(\frac{1}{\frac{c_n}{\Delta_n}+(1-c_n)\left(\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}+\frac{\Delta_n-1}{\Delta_n}\right)}-\frac{1}{\frac{c_n}{\Delta_n}+(1-c_n)\frac{\Delta_n-1}{\Delta_n}}\right)\\
&= \sqrt{n}\frac{1-c_n}{1-c_n-\frac{1-2c_n}{\Delta_n}}\,(1-\frac{1}{\Delta_n})\,\left(\frac{1}{1+\frac{1-c_n}{1-c_n-\frac{1-2c_n}{\Delta_n}}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}-1\right)\\
&= -\frac{(1-c_n)^2}{(1-c_n-\frac{1-2c_n}{\Delta_n})^2}\,(1-\frac{1}{\Delta_n})\,\frac{\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
I_n + II_n &= \sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\frac{c_n(1-c_n)}{(1-c_n-\frac{1-2c_n}{\Delta_n})^2}\,\frac{1}{\Delta_n}\,\frac{1}{1+\frac{1-c_n}{\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n})}\sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}}\\
&= \sqrt{n}\frac{\hat{R}_{\mathbf{b}_n}-\Delta_n+1}{\Delta_n}\frac{c_n(1-c_n)\Delta_n}{(c_n+(\Delta_n-1)(1-c_n))^2}\,(1+o_p(1))\\
&\overset{d}{\approx} \mathcal{N}\left(0, 2\frac{c_n^3(1-c_n)\Delta_n^2}{(c_n+(\Delta_n-1)(1-c_n))^4}\right)
\end{aligned}$$

if $\sqrt{n}(1-c_n-\frac{1-2c_n}{\Delta_n}) \to \infty$ as $n \to \infty$. Since $\mathbf{b}_n'\mathbf{\Sigma}_n\mathbf{b}_n \geq \min_{\mathbf{w}} \mathbf{w}'\mathbf{\Sigma}_n\mathbf{w} = \frac{1}{\mathbf{1}_n'\mathbf{\Sigma}_n^{-1}\mathbf{1}_n}$, it holds that $\Delta_n \geq 1$, and, thus, this condition is fulfilled. □